# Project-and-Lift for the Perspective Reformulation: How Serendipity Brought Us to a Free Lunch

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# Outline

# The Hydro-thermal Unit Commitment problem

- Set $P$ of thermal units and $H$ of hydro (cascade) units
- Discretized time horizon $\mathcal{T}$, energy demand $\bar{d}_t$ for $t \in \mathcal{T}$
- Complicated technical constraints
- Operate the set of available units over $\mathcal{T}$ so as to satisfy demand
- A (not particularly smart) MIQP model of a "simple" version
  - $u^i_t \in \{0, 1\}$: ON/OFF state of thermal unit $i \in P$
  - $p^i_t \in \mathbb{R}_+$: power level of thermal unit $i \in P$
  - $q^j_t \in \mathbb{R}_+$: water discharge for hydro unit $j \in H(h)$ for cascade $h \in H$
- Objective function:

$$f(p, u) = \sum_{i \in P} \left( s^i(u^i) + \sum_{t \in \mathcal{T}} \left( a^i_t (p^i_t)^2 + b^i_t p^i_t + c^i_t u^i_t \right) \right) \qquad (1)$$

  - nonlinear convex energy cost ($a^i_t > 0$), fixed costs
  - time-dependent start-up costs $s^i(u^i)$ (only a few extra constraints and continuous variables with nifty formulation[1])

[1] Nowak, Römisch "Stochastic Lagrangian Relaxation Applied to Power Scheduling in a Hydro-thermal System Under Uncertainty", *Annals of Operations Research*, 2000

# A MIQP formulation of UC

- Thermal units:

$$\bar{p}^i_{min} u^i_t \leq p^i_t \leq \bar{p}^i_{max} u^i_t \qquad t \in \mathcal{T} \quad (2)$$

$$p^i_t \leq p^i_{t-1} + u^i_{t-1}\Delta^i_+ + (1 - u^i_{t-1})\bar{l}^i \qquad t \in \mathcal{T} \quad (3)$$

$$p^i_{t-1} \leq p^i_t + u^i_t\Delta^i_- + (1 - u^i_t)\bar{u}^i \qquad t \in \mathcal{T} \quad (4)$$

$$u^i_t \leq 1 - u^i_{r-1} + u^i_r \qquad t \in \mathcal{T},\ r \in [t - \tau^i_+, t-1] \quad (5)$$

$$u^i_t \geq 1 - u^i_{r-1} - u^i_r \qquad t \in \mathcal{T},\ r \in [t - \tau^i_-, t-1] \quad (6)$$

- Hydro units:

$$0 \leq q^j_t \leq \bar{q}^j_{max} \qquad t \in \mathcal{T} \quad (7)$$

$$\bar{v}^j_{min} \leq v^j_t \leq \bar{v}^j_{max} \qquad t \in \mathcal{T} \quad (8)$$

$$v^j_t - v^j_{t-1} = \bar{w}^j_t - w^j_t - q^j_t + \textstyle\sum_{k \in \mathcal{S}(j)} \left( q^k_{t-t_{kj}} + w^k_{t-t_{kj}} \right) \qquad t \in \mathcal{T} \quad (9)$$

- Demand satisfaction ($\alpha^j$ = constant power-to-discharged water):

$$\textstyle\sum_{i \in P} p^i_t + \sum_{h \in H} \sum_{j \in H(h)} \alpha^j q^j_t = \bar{d}_t \qquad t \in \mathcal{T} \quad (10)$$

# Traditional solution approaches

- Large-scale (large $|P|$, $|H|$, $|\mathcal{T}|$) Mixed-Integer Quadratic Program to be solved in a few minutes
- Traditionally intractable for general-purpose MIQP/MILP solvers
- Traditional alternative: Lagrangian Relaxation of demand constraints (10), Lagrangian heuristic[2]
- Results actually quite good, especially if compared with `Cplex`
- A pesky Referee's comment: may the problem for Cplex be the Quadratic part? If so, piecewise-linearize $f$[3]
- Should this work? On the outset, we didn't see why ...

[2] F., Gentile, Lacalandra "Solving Unit Commitment Problems with General Ramp Contraints", *IJEPES*, 2008

[3] Carrión, Arroyo "A Computationally Efficient Mixed-integer Linear Formulation for the Thermal Unit Commitment Problem" *IEEE Transactions on Power Systems*, 2006

# Results of the MILP formulation

... but it did, big times!

| | MIQP | | | MILP | | | | |
|---|---|---|---|---|---|---|---|---|
| | first | best | gap | time | gap | ftime | fgap | nodes |
| 20 | 24 | 2229 | 0.29 | 3.72 | 0.36 | | 1.00 | 0 |
| 50 | 249 | 1491 | 0.22 | 21.93 | 0.21 | 15.98 | 0.36 | 0 |
| 75 | 447 | 1514 | 0.10 | 56.31 | 0.20 | 47.08 | 1.62 | 10 |
| 100 | 940 | 2327 | 0.13 | 94.09 | 0.17 | 69.75 | 2.18 | 16 |
| 150 | 2348 | 2483 | 0.24(1) | 218.69 | 0.12 | 177.35 | 6.58 | 16 |
| 200 | 3600 | 3600 | * (5) | 267.78 | 0.09 | 247.12 | 1.85 | 6 |

- Stopping tolerance at 0.5% (and invalid lower bound)
- Again, inherent gap vastly worse (and invalid anyway)
- Most of the difference is in the heuristic, as the LB is weaker (. . . )

# Comparing MILP and LR

| | | RCDP | | | Cplex MILP | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $h$ | time | gap | iter | time | gap | ftime | fgap | nodes | LPs |
| 10 | 0 | 0.75 | 0.99 | 187 | 0.95 | 0.33 | | 1.18 | 0 | 23 |
| 20 | 0 | 1.83 | 0.46 | 189 | 3.72 | 0.36 | | 1.00 | 0 | 23 |
| 50 | 0 | 4.84 | 0.28 | 195 | 21.93 | 0.21 | 15.98 | 0.36 | 0 | 25 |
| 75 | 0 | 9.41 | 0.34 | 206 | 56.31 | 0.20 | 47.08 | 1.62 | 10 | 59 |
| 100 | 0 | 14.74 | 0.33 | 213 | 94.09 | 0.17 | 69.75 | 2.18 | 16 | 76 |
| 150 | 0 | 21.20 | 0.17 | 277 | 218.69 | 0.12 | 177.35 | 6.58 | 16 | 115 |
| 200 | 0 | 34.80 | 0.09 | 317 | 267.78 | 0.09 | 247.12 | 1.85 | 6 | 87 |
| 20 | 10 | 1.76 | 0.39 | 170 | 93.53 | 0.21 | | 0.59 | 140 | 258 |
| 50 | 20 | 6.36 | 0.06 | 160 | 17.98 | 0.06 | 17.98 | 0.06 | 0 | 60 |
| 75 | 35 | 15.01 | 0.04 | 198 | 96.86 | 0.11 | 96.86 | 0.11 | 170 | 300 |
| 100 | 50 | 24.74 | 0.04 | 209 | 130.86 | 0.06 | 130.86 | 0.06 | 180 | 266 |
| 150 | 75 | 37.41 | 0.02 | 189 | 467.62 | 0.06 | 467.62 | 0.06 | 300 | 554 |
| 200 | 100 | 50.91 | 0.01 | 175 | 427.71 | 0.05 | 427.71 | 0.05 | 205 | 321 |

- Cplex primal heuristic impressively effective despite the LB being much worse . . . or is it?

# Here Comes the Serendipity Moment

- Testing the MILP formulation to appease the pesky Referee (who happens to be right, albeit for the wrong reason: bad enough already)
- The LB of the MILP must be lower than that of the MIQP (which is lower than that of the LR)

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  and instead is a (n involuntary) reformulation which improves the LB!
- After lots of head scratching, here's what had happened

# The general MINLP framework

- Convex function $f$, Mixed-Integer NonLinear Program fragment

$$\min \{\ f(p) + cu\ :\ Ap \leq bu\ ,\ u \in \{0,1\}\ \} \tag{11}$$

$p \in \mathcal{P} = \{\ p \in \mathbb{R}^n\ :\ Ap \leq b\ \}$ compact $\equiv \{\ p\ :\ Ap \leq 0\ \} = \{0\}$

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- Equivalently, minimize the nonconvex function

$$f(p,u) = \begin{cases} 0 & \text{if } u = 0 \text{ and } p = 0 \\ f(p) + c & \text{if } u = 1 \text{ and } Ap \leq b \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

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- Best possible convex relaxation of (11): use the convex envelope[4]

$$h(p,u) = \begin{cases} 0 & \text{if } p = 0 \text{ and } u = 0, \\ uf(p/u) + cu & \text{if } Ap \leq bu, \;\; u \in (0,1], \\ +\infty & \text{otherwise.} \end{cases} \tag{13}$$

(convex function minorizing $f(p,u)$ with smallest possible epigraph)

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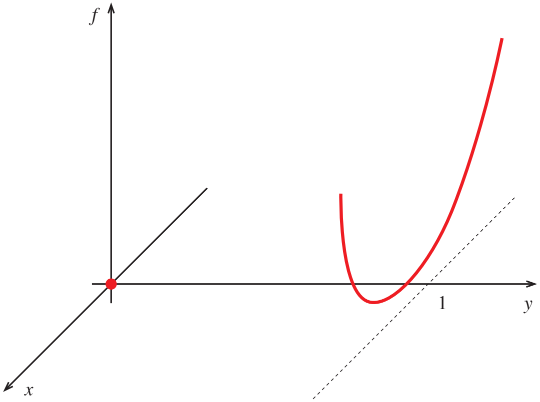
# The Perspective what?

- $h(p, u)$ is a section of the perspective function $f(x, \lambda) = \lambda f(x/\lambda)$



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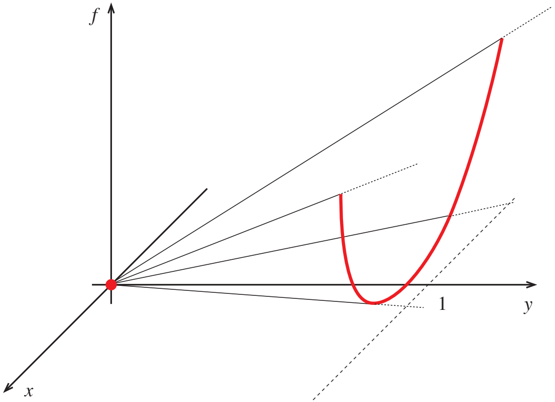

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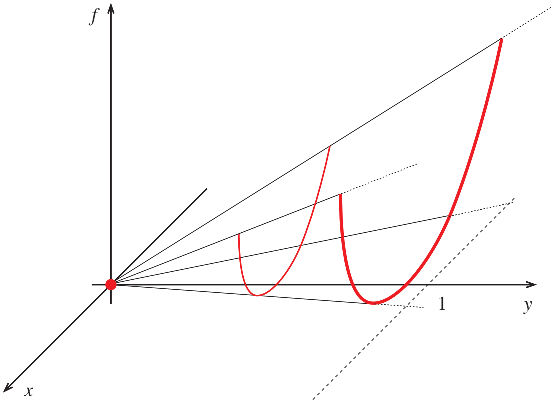

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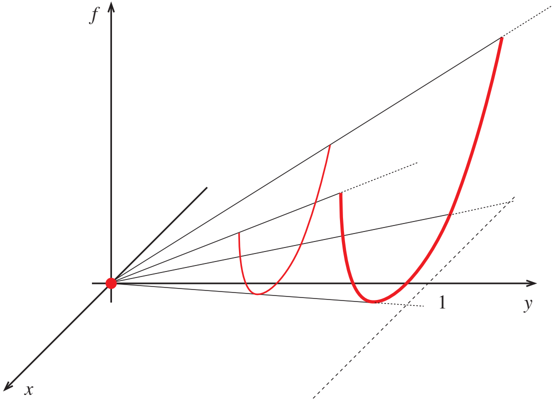


- $h(p, u)$ convex but much more nonlinear than $f(p) + cu$

example: $f(p) = ap^2 + bp \quad \Longrightarrow \quad h(p, u) = (a/u)p^2 + bp + cu$

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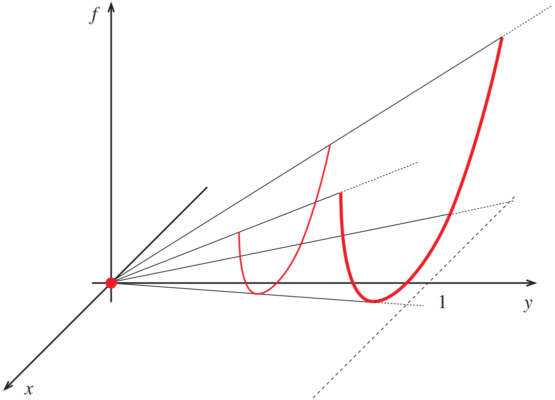


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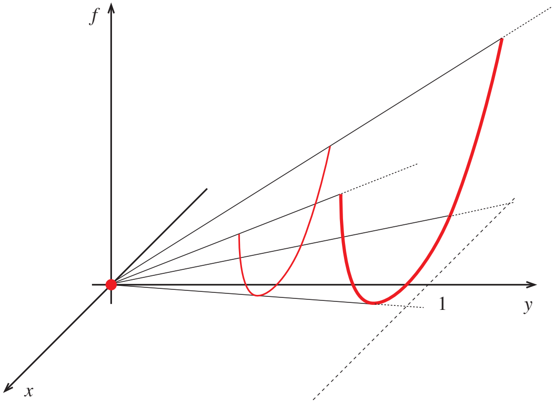


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notes: I) $a/u > a$ for $u < 1$; II) for $a = 0$ nothing happens

# The Perspective Relaxation (Reformulation)

- A convex, continuous, more nonlinear program

$$\min \left\{ \, uf(p/u) + cu \;:\; Ap \leq bu \;,\; u \in \{0,1\} \, \right\} \tag{14}$$

$u \in \{0,1\}^n \Longrightarrow$ a reformulation! (if $0f(0/0) = 0 \ldots$)

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- Every convex function is the supremum of its affine minorants

$(v, p, u) \in epi\ h \iff Ap \leq bu$, $u \in [0,1]$, and $\forall \bar{p} \in \mathcal{P}$

$$v \geq f(\bar{p}) + c + [s \; , \; c + f(\bar{p}) - s\bar{p}] \begin{bmatrix} p - \bar{p} \\ u - 1 \end{bmatrix} \qquad \forall s \in \partial f(\bar{p}) \tag{15}$$

(infinitely many inequalities, at least one for each $\bar{p} \in \mathcal{P}$)

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(infinitely many inequalities, at least one for each $\bar{p} \in \mathcal{P}$)

- The quadratic case: Perspective Cuts

$$v \geq (2a\bar{p} + b)p + (c - a\bar{p}^2)u \qquad \forall \bar{p} \in [p_{min}, p_{max}] \tag{16}$$

(we happened to have inadvertently slipped in a pair of these)

# Impact of the Perspective Relaxation (P/C)

| P/C | | | | CPLEX | | | | |
|---|---|---|---|---|---|---|---|---|
| r.time | r.gap | time | nodes | r.time | r.gap | time | nodes | gap |
| 4.17 | 0.28 | 15.61 | 3 | 1.41 | 2.36 | 10000 | 264179 | 1.27 |
| 4.29 | 0.13 | 4.53 | 1 | 1.80 | 0.49 | 62 | 1205 | - |
| 2.07 | 0.69 | 178.12 | 136 | 0.85 | 1.24 | 216 | 4083 | - |
| 8.64 | 0.28 | 37.14 | 4 | 1.61 | 2.40 | 10000 | 331732 | 1.43 |
| 8.42 | 0.20 | 23.75 | 2 | 1.71 | 1.63 | 10000 | 245582 | 0.87 |
| 6.71 | 0.24 | 12.59 | 2 | 1.58 | 1.37 | 10000 | 268516 | 0.73 |
| 4.83 | 0.28 | 12.71 | 3 | 0.87 | 2.23 | 10000 | 475400 | 1.45 |
| 5.97 | 0.18 | 19.35 | 3 | 1.74 | 1.06 | 6137 | 189898 | - |
| 6.73 | 0.23 | 44.35 | 44 | 1.55 | 2.60 | 10000 | 337915 | 1.69 |
| 7.96 | 0.26 | 141.69 | 73 | 1.64 | 2.28 | 10000 | 286651 | 1.02 |
| 5.98 | 0.28 | 48.98 | 57 | 1.48 | 1.77 | 7642 | 240516 | 0.85 |

- Root gap node greatly reduced at a small expense in running time
- Small instances ($p = 20$), no ramp constraints, gap $= 0.1\%$
- Nice thing is: you only need a `cutcallback`

# Extension to nonseparable functions

- Another pesky referee wanted results on another problem than UC
- Mean-Variance problem with min and max buy-in thresholds

$$\min \left\{ x^T Q x \;\middle|\; \begin{array}{l} ex = 1 \ , \ \ \mu x \geq \rho \ , \\ l_i y_i \leq x_i \leq u_i y_i \ , \ y_i \in \{0,1\} \ \ i = 1, \ldots, n \end{array} \right\}$$

$\mu$ = expected return, $Q$ = covariance matrix, $\rho$ = desired return

- But $f$ is nonseparable, so Perspective Reformulation not applicable

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  $\mu =$ expected return, $Q =$ covariance matrix, $\rho =$ desired return

- But $f$ is nonseparable, so Perspective Reformulation not applicable
- Dirty trick: choose $D \succeq 0$ diagonal s.t. $R = Q - D \succeq 0$

$$\min \left\{ x^T D x + z^T R z \;\middle|\; \begin{array}{l} ex = 1 \;\;,\;\; \mu x \geq \rho \;\;,\;\; z = x \\ l_i y_i \leq x_i \leq u_i y_i \;,\; y_i \in \{0,1\} \;\; i = 1, \ldots, n \end{array} \right\}$$

  Move nonseparability to new variables $p$, let $D$ "as large as possible"

- Trivial choice: $\mathrm{D} = \lambda_{min}(Q) I$

# The instances

- 30 randomly-generated instances for each $n \in \{200, 300, 400\}$
- $\mu_i \in [0.002, 0.01]$, $l_i \in [0.075, 0.125]$, $u_i \in [0.375, 0.425]$ (uniformly)
- $Q$ = well-known random generator of [Pardalos, Rodgers '90]
- Effectiveness of Perspective Relaxation heavily impacted by

$$S = \text{average} \left\{ \frac{Q_{ii} - \sum_{j \neq i} |Q_{ij}|}{Q_{ii}} \ : \ i = 1, \ldots, n \right\}$$

  (dominance index)

- For each $n$, three classes of instances (10 each):
  - "$+$" instances, $S \approx 0.6$ (diagonally dominant)
  - "0" instances, $S \approx 0$ (diagonally quasi-dominant)
  - "$-$" instances, $S \approx -0.5$ (not diagonally dominant)
- Available at `http://www.di.unipi.it/optimize/Data`

# Results of P/C

| | P/C | | | | Cplex | | | |
|---|---|---|---|---|---|---|---|---|
| | time | nodes | d.gap | r.gap | nodes | p.gap | d.gap | r.gap |
| $200^+$ | 904 | 7.7e+4 | | 6.48 | 1.9e+7 | 0.14 | 45.33 | 85.63 |
| $200^0$ | 320 | 2.8e+4 | | 6.10 | 8.5e+6 | 0.38 | 51.27 | 84.47 |
| $200^-$ | 3306 | 2.6e+5 | 0.02 | 6.69 | 8.9e+6 | 0.24 | 42.09 | 78.88 |
| $300^+$ | 2061 | 9.3e+4 | | 5.62 | 4.0e+6 | 0.41 | 64.68 | 92.01 |
| $300^0$ | 1715 | 7.1e+4 | | 6.28 | 3.6e+6 | 0.43 | 59.91 | 87.87 |
| $300^-$ | 2797 | 9.4e+4 | 0.05 | 7.04 | 3.0e+6 | 0.53 | 45.11 | 78.77 |
| $400^+$ | 4756 | 1.1e+5 | 0.10 | 6.15 | 1.9e+6 | 1.03 | 61.47 | 89.06 |
| $400^0$ | 7421 | 1.6e+5 | 0.16 | 6.53 | 1.5e+6 | 1.18 | 68.68 | 90.03 |
| $400^-$ | 6901 | 1.4e+5 | 0.36 | 6.49 | 1.5e+6 | 1.60 | 65.88 | 88.47 |

- Root node gap divided by 10: $< 8\%$ w.r.t. $> 80\%$
- Effectiveness worsens as $Q$ less dominant, could not even solve all 200 instances (though much better than Cplex: none solved in 10000s)
- Perhaps a better $D$ could help?

# Choosing $D$ via SDP

- Assuming $tr(D)$ the relevant metric, the "largest" $D$ solves

$$\max \left\{ \sum_{i=1}^{n} d_i \; : \; Q - \sum_{i=1}^{n} d_i(e_i e_i^T) \succeq 0 \; , \; d \geq 0 \right\}$$

$$\min \left\{ tr(QX) \; : \; diag(X) \geq e \; , \; X \succeq 0 \right\}$$

  dual pair of SemiDefinite (= convex = easy) Problems

- Several efficient, open-source SDP codes

- Interesting relaxation: removing $d \geq 0$ in the primal gives

$$\min \left\{ tr(QX) \; : \; diag(X) = e \; , \; X \succeq 0 \right\}$$

  - $d^* > 0$ anyway in all our tests
  - most often faster to solve in practice by all codes
  - *constant trace* = max eigenvalue problem, specialized approaches

- Is the SDP running time low? Is this worth?

# Comparison of SDP codes

| | $d_{max}$ | $d_{min}$ | $d_{avg}$ | ME | CSDP | | DSDP | | SDPA | | SDPLR | | SB |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\geq$ | $=$ | $\geq$ | $=$ | $\geq$ | $=$ | $\geq$ | $=$ | $=$ |
| $200^+$ | 1.96 | 0.97 | 1.47 | 0.13 | 3.12 | 2.98 | 1.86 | 0.10 | 1.81 | 0.29 | 3.71 | 2.23 | 23.77 |
| $200^0$ | 1.93 | 0.90 | 1.41 | 0.13 | 3.03 | 2.99 | 1.87 | 0.10 | 1.68 | 0.29 | 3.72 | 2.79 | 16.39 |
| $200^-$ | 1.86 | 0.87 | 1.37 | 0.13 | 3.00 | 2.95 | 1.86 | 0.10 | 1.62 | 0.40 | 2.30 | 2.19 | 16.58 |
| $300^+$ | 1.97 | 0.97 | 1.47 | 0.23 | 10.54 | 9.84 | 4.92 | 0.26 | 5.33 | 0.73 | 13.20 | 5.02 | 69.13 |
| $300^0$ | 1.93 | 0.91 | 1.42 | 0.23 | 10.91 | 9.55 | 4.99 | 0.26 | 4.97 | 0.71 | 8.58 | 9.08 | 46.01 |
| $300^-$ | 1.69 | 0.89 | 1.29 | 0.23 | 10.91 | 9.62 | 5.10 | 0.26 | 5.11 | 0.72 | 5.67 | 5.53 | 41.82 |
| $400^+$ | 1.98 | 0.97 | 1.47 | 0.39 | 31.03 | 29.28 | 10.56 | 0.52 | 5.02 | 1.40 | 17.48 | 21.60 | 146.07 |
| $400^0$ | 1.93 | 0.93 | 1.43 | 0.39 | 37.24 | 31.27 | 10.86 | 0.52 | 11.46 | 1.37 | 21.80 | 11.93 | 94.62 |
| $400^-$ | 1.87 | 0.89 | 1.38 | 0.39 | 36.77 | 31.61 | 10.75 | 0.52 | 11.10 | 1.38 | 15.10 | 21.11 | 90.07 |

- On average 50% better than $\lambda_{min}$, worst case $\approx$ few % worse
- Results getting worse as $Q$ less diagonally dominant
- Times not much worse using right code and model
- Is it worth?

# Impact on the B&C

| | SDP | | | | ME | | | |
|---|---|---|---|---|---|---|---|---|
| | time | nodes | d.gap | r.gap | time | nodes | d.gap | r.gap |
| $200^+$ | 164 | 1.2e+4 | | 1.14 | 904 | 7.7e+4 | | 6.48 |
| $200^0$ | 161 | 1.1e+4 | | 2.14 | 320 | 2.8e+4 | | 6.10 |
| $200^-$ | 1902 | 1.3e+5 | | 3.65 | 3306 | 2.6e+5 | 0.02 | 6.69 |
| $300^+$ | 818 | 2.9e+4 | | 4.54 | 2061 | 9.3e+4 | | 5.62 |
| $300^0$ | 856 | 2.7e+4 | | 1.97 | 1715 | 7.1e+4 | | 6.28 |
| $300^-$ | 1709 | 5.2e+4 | | 2.68 | 2797 | 9.4e+4 | 0.05 | 7.04 |
| $400^+$ | 2264 | 7.0e+4 | | 4.79 | 4756 | 1.1e+5 | 0.10 | 6.15 |
| $400^0$ | 4378 | 7.2e+4 | 0.10 | 2.29 | 7421 | 1.6e+5 | 0.16 | 6.53 |
| $400^-$ | 6311 | 1.0e+5 | 0.23 | 3.06 | 6901 | 1.4e+5 | 0.36 | 6.49 |

- root node gap halved+ w.r.t. ME
- All instances up to $n = 300$ solved to optimality within 10000s
- Effectiveness still worsens as $Q$ less dominant, can't solve a few $400^-$

# An Alternative: The Conic Program Reformulation

- $\lambda f(x/\lambda)$ is SOCP-representable if $f$ is [Ben Tal '02]

[5] F., Gentile "A Computational Comparison of [. . . ]: SOCP vs. Cutting Planes" *Op. Res. Letters*, 2009

# An Alternative: The Conic Program Reformulation

- $\lambda f(x/\lambda)$ is SOCP-representable if $f$ is [Ben Tal '02]
- Some pesky people insisted in ignoring (16) and using instead

$$\min \left\{ t + bp + cu : \sqrt{ap^2} \leq tu \ \ldots \ u \in \{0,1\} \right\}$$

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| | | P/C | | | | | CP | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $h$ | gap | nodes | LPs | time | t/LP | gap | nodes | CPs | time | t/CP |
| 10 | 0 | | 4.3e2 | 7.8e2 | 14 | 0.018 | | 5.8e2 | 1.0e3 | 20 | 0.021 |
| 20 | 0 | | 5.0e4 | 5.8e4 | 6805 | 0.094 | | 6.6e4 | 7.5e4 | 13392 | 0.145 |
| 50 | 0 | 0.08 | 1.7e5 | 2.1e5 | 86400 | 0.421 | 0.08 | 9.1e4 | 1.1e5 | 86400 | 0.781 |
| 20 | 10 | | 1.1e4 | 1.3e4 | 161 | 0.014 | | 1.4e4 | 1.8e4 | 626 | 1.937 |
| 50 | 20 | | 5.5e5 | 6.6e5 | 29874 | 0.037 | 0.00 | 5.0e5 | 6.1e5 | 86400 | 0.460 |
| 75 | 35 | 0.01 | 8.5e5 | 1.0e6 | 73076 | 0.073 | 0.01 | 1.8e5 | 2.2e5 | 86400 | 0.314 |

... but it's not a great idea (24h = very unrealistic, gap = 0.01%)[5]

[5] F., Gentile "A Computational Comparison of [...]: SOCP vs. Cutting Planes" *Op. Res. Letters*, 2009

# Comparing CP with P/C on MV

| | P/C | | | | CP | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | nodes | QPs | time | t/QP | nodes | CPs | time | t/CP | gap |
| $200^+$ | 1.9e4 | 1.9e4 | 194 | 0.0008 | 9.2e3 | 1.1e3 | 17961 | 1.578 | 0.15(1) |
| $200^0$ | 1.7e4 | 1.8e4 | 90 | 0.0007 | 2.7e4 | 3.2e4 | 30785 | 1.648 | 0.32(2) |
| $200^-$ | 1.2e5 | 1.3e5 | 835 | 0.0006 | 1.6e4 | 1.9e5 | 55144 | 1.719 | 1.02(5) |
| $300^+$ | 3.4e4 | 3.5e4 | 433 | 0.0014 | 1.1e4 | 1.4e4 | 72075 | 8.334 | 0.58(7) |
| $300^0$ | 3.1e5 | 3.3e4 | 378 | 0.0019 | 1.0e4 | 1.3e4 | 59591 | 4.464 | 0.53(6) |
| $300^-$ | 5.5e5 | 5.8e4 | 654 | 0.0014 | 1.1e4 | 1.3e4 | 66863 | 5.272 | 0.81(7) |
| $400^+$ | 7.9e4 | 8.2e4 | 2066 | 0.0032 | 4.7e3 | 5.9e3 | 61810 | 10.397 | 1.01(6) |
| $400^0$ | 2.3e5 | 2.4e5 | 3974 | 0.0020 | 6.1e3 | 7.6e3 | 83782 | 10.588 | 1.79(9) |
| $400^-$ | 3.3e5 | 3.4e5 | 8092 | 0.0026 | 6.3e3 | 7.9e3 | 80382 | 10.764 | 2.71(8) |

- Same SDP solved for CP and P/C
- QPs awesomely faster than CPs to solve ("quadratic simplex") even invoking the built-in SOCP linearization (???)
- Faster bound $\Longrightarrow$ more nodes $\Longrightarrow$ faster convergence
- Best case (very unstructured problem)

# Alternatives to P/C and CP to solve the PR

- We tried quite a few (Newton, ...) before finding CP

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# Alternatives to P/C and CP to solve the PR

- We tried quite a few (Newton, ...) before finding CP
- The (failed) Newton idea eventually led us to projecting
- Only works under strong assumptions (often true):

  A1 $p$ is a single variable, $\bar{p}_{min} \geq 0$

  A2 $f$ is quadratic

  A3 there are no constraints linking different $u$

- Basic idea: recast the PR as

$$\min \{ \, z(p) \; : \; p \in [0, \bar{p}_{max}] \, \}$$

where $z(p)$ (partial minimization of a convex function $\Rightarrow$ convex) is

$$z(p) = bp + \min_u \{ \, ap^2/u + cu \; : \; u\bar{p}_{min} \leq p \leq u\bar{p}_{max} \; , \; p \in [0,1] \, \}$$

(intuition: the projection will be "less nonlinear")

# Derivation

- Algebraic characterization of $z(p)$ out of optimal solution $u^*(p)$[6]
- In turn, $u^*(p)$ out of unconstrained minimizer $\tilde{u}(p)$, i.e., solution to
$$\frac{\partial h(p,u)}{\partial u} = c - ap^2/p^2 = 0$$
(if any)
    - if $\tilde{u}(p)$ is feasible, then $u^*(p) = \tilde{u}(p)$
    - otherwise, $u^*(p)$ is the projection of $\tilde{u}(p)$ over the feasible region (easy because the feasible region is very simple)
    - note: $ap^2/u^2 \geq 0$
- Basically, a few algebraic computations depending on $a$, $c$, ...

[6] F., Gentile, Grande, Pacifici "Projected Perspective Reformulations with Applications in Design Problems" *Operations Research* 2010

# The form of $z(p)$

1) $c \leq 0 \Longrightarrow \tilde{u}(p)$ undefined $\Longrightarrow u^*(p) = 1 \Longrightarrow$

$$\boxed{z(p) = ap^2 + bp + c}$$

2) $c > 0 \Longrightarrow \tilde{u}(p) = p\sqrt{a/c}$

2.1) $\tilde{u} \leq p/\bar{p}_{max} \Longleftrightarrow \bar{p}_{max} \leq \sqrt{c/a} \Longleftrightarrow u^*(p) = p/\bar{p}_{max} \Longrightarrow$

$$\boxed{z(p) = (\ b + a\bar{p}_{max} + c/\bar{p}_{max}\ )p}$$

2.2) $0 \geq \tilde{u}(p) \geq p/\bar{p}_{max} \Longleftrightarrow \bar{p}_{max} \geq \sqrt{c/a}\ (\geq \bar{p}_{min})$.

- $(\bar{p}_{max} \geq)\ p \geq \sqrt{c/a}\ (\geq 0) \Longrightarrow \tilde{u}(p) \geq 1 \Longrightarrow u^*(p) = 1$;
- $0 \leq p \leq \sqrt{c/a}\ (\leq \bar{p}_{max}) \Longrightarrow \tilde{u}(p) \leq 1 \Longrightarrow u^*(p) = \tilde{u}(p)$.

$$\Longrightarrow \boxed{z(p) = \begin{cases} (\ b + 2\sqrt{ac}\ )p & \text{if } 0 \leq p \leq \sqrt{c/a} \\ ap^2 + bp + c & \text{if } \sqrt{c/a} \leq p \leq \bar{p}_{max} \end{cases}}$$

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- $z(p)$ convex differentiable piecewise-quadratic with $\leq 2$ pieces

# Application: Quadratic (1-Commodity) Network Design

- Directed graph $G = (N, A)$, deficit $b_i$ for $i \in N$, arc capacity $\bar{p}_{ij}$ with fixed-charge cost $c_{ij} > 0$, quadratic routing cost $b_{ij} p_{ij} + a_{ij} p_{ij}^2$

$$
\begin{aligned}
\min \quad & \textstyle\sum_{(i,j) \in A} c_{ij} u_{ij} + b_{ij} p_{ij} + a_{ij} p_{ij}^2 \\
& \textstyle\sum_{(j,i) \in A} p_{ji} - \sum_{(i,j) \in A} p_{ij} = b_i \quad i \in N \\
& 0 \leq p_{ij} \leq \bar{p}_{ij} u_{ij} \;\; , \;\; u_{ij} \in \{0, 1\} \quad (i,j) \in A
\end{aligned}
\tag{17}
$$

- In continuous relaxation, $u_{ij} = p_{ij} / \bar{p}_{ij} \Longrightarrow$ a quadratic flow problem
- Very weak bound, PR improves it but destroys flow structure

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$$\begin{array}{ll} \min & \sum_{(i,j) \in A} c_{ij} u_{ij} + b_{ij} p_{ij} + a_{ij} p_{ij}^2 \\ & \sum_{(j,i) \in A} p_{ji} - \sum_{(i,j) \in A} p_{ij} = b_i \quad i \in N \\ & 0 \leq p_{ij} \leq \bar{p}_{ij} u_{ij} \ , \ \ u_{ij} \in \{0,1\} \quad (i,j) \in A \end{array} \tag{17}$$

- In continuous relaxation, $u_{ij} = p_{ij} / \bar{p}_{ij} \Longrightarrow$ a quadratic flow problem
- Very weak bound, PR improves it but destroys flow structure
- P²R: Separable Convex Quadratic MCF on $G' = (N, A')$ with $|A'| \leq 2|A|$ (same nodes, duplicated arcs)

$$\begin{array}{lll} \min & \sum_{(i,j) \in A'} b'_{ij} r_{ij} + a'_{ij} r_{ij}^2 & \\ & \sum_{(j,i) \in A'} r_{ji} - \sum_{(i,j) \in A'} r_{ij} = b_i & i \in N \\ & 0 \leq r_{ij} \leq \bar{r}_{ij} & (i,j) \in A' \end{array}$$

# Computational results

- Randomly generated Network Design instances:
  - standard DIMACS random MCF problem generator (`netgen`)
  - linear costs randomly generated in a given interval
  - quadratic/fixed costs randomly generated with respect to linear costs (two different ways, "h" – high and "l" – low)
- Not as difficult as expected (many solved at root node)
- 2-pieces linear/quadratic $P^2/R$ solved with `CPLEX-qpopt`
  $\Longrightarrow$ not really a specialized algorithm
- Using "true" MCF solver possible with further piecewise-linearization but static version not competitive (dynamic may be)
- $P^2/R$ very efficient (but helped by few branching/cuts)
- Non P/R B&C and Conic Program formulation of P/R much slower

# Network Design — the table

| name | time | nodes | t/n | time | nodes | t/n | gap |
|---|---|---|---|---|---|---|---|
| | $P^2/R$ | | | CPLEX | | | |
| 2000-h-h | 0.10 | 1 | 0.10 | 690.09 | 101868 | 0.11 | 0.00 |
| 2000-h-l | 45.42 | 278 | 1.10 | 1031.75 | 141485 | 0.01 | 0.06 |
| 2000-l-h | 0.09 | 1 | 0.09 | 858.22 | 131954 | 0.03 | 0.00 |
| 2000-l-l | 8.78 | 63 | 0.10 | 1036.79 | 140877 | 0.01 | 0.04 |
| 3000-h-h | 0.15 | 1 | 0.15 | 1041.96 | 88541 | 0.01 | 0.00 |
| 3000-h-l | 71.02 | 269 | 0.17 | 1051.93 | 73591 | 0.01 | 0.12 |
| 3000-l-h | 0.15 | 1 | 0.15 | 988.74 | 89209 | 0.12 | 0.00 |
| 3000-l-l | 19.05 | 79 | 0.16 | 1062.45 | 85878 | 0.01 | 0.04 |
| | P/C | | | CP | | | |
| 2000-h-h | 57.09 | 7 | 13.84 | 895.70 | 8 | 207.60 | 0.01 |
| 2000-h-l | 51.60 | 348 | 0.72 | 252.98 | 36 | 27.65 | 0.00 |
| 2000-l-h | 42.3 | 6 | 16.57 | 525.35 | 9 | 63.35 | 0.00 |
| 2000-l-l | 20.60 | 131 | 0.51 | 252.82 | 193 | 40.02 | 0.00 |
| 3000-h-h | 117.30 | 11 | 18.90 | 564.41 | 2 | 407.97 | 0.01 |
| 3000-h-l | 140.47 | 584 | 1.39 | 366.95 | 27 | 36.76 | 0.00 |
| 3000-l-h | 101.18 | 12 | 12.01 | 372.16 | 4 | 89.53 | 0.01 |
| 3000-l-l | 45.43 | 153 | 0.89 | 292.41 | 83 | 62.39 | 0.00 |

# Approximated P$^2$R for Controlled Tabular Adjustment

- Statistical table, $\mathcal{N}$ cells, data $d$ satisfies system $Ad = b$
- Find some perturbation $z$ such that:
  - $A(d + z) = b \iff Az = 0$
  - for sensitive cells $i \in \mathcal{S} \subseteq \mathcal{N}$, either $z_i \geq u_i$ or $z_i \leq -l_i$
  - minimize some (weighted) $||z||$
- Positive and negative perturbation variables: $z_i = z_i^+ - z_i^-$ for $i \in \mathcal{N}$
- Obvious MIQP formulation (given weights $w$)

$$
\begin{array}{lll}
\min & \sum_{i \in \mathcal{N}} w_i (z_i^+ - z_i^-)^2 & \\
 & A(z^+ - z^-) = 0 & \\
 & 0 \leq z_i^+ \leq \bar{u}_i \ , \ 0 \leq z_i^- \leq \bar{l}_i & i \in \mathcal{U} = \mathcal{N} \setminus \mathcal{S} \\
 & \left.\begin{array}{lcl} u_i y_i^+ \leq z_i^+ \leq \bar{u}_i y_i^+ & , & l_i y_i^- \leq z_i^- \leq \bar{l}_i y_i^- \\ y_i^+ + y_i^- = 1 & , & y_i^\pm \in \{0, 1\} \end{array}\right\} & i \in \mathcal{S}
\end{array}
$$

- Disjunctive constraints, mutually exclusive semicontinuous variables

# Perspective Reformulation

- PR not directly applicable: $w_i(z_i^+ - z_i^-)^2 \Longrightarrow w_i\big((z_i^+)^2 + (z_i^-)^2\big)$

$$\min \textstyle\sum_{i\in\mathcal{U}} \; w_i\big(\, (z_i^+)^2 + (z_i^-)^2 \,\big) + \; \sum_{i\in\mathcal{S}} \; w_i \left( \frac{(z_i^+)^2}{y_i^+} + \frac{(z_i^-)^2}{y_i^-} \right)$$

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- (Trivial) theorem: substitute $y_i = y_i^+$ and $1 - y_i = y_i^-$

$$f(\,z^+\,,\,z^-\,,\,y\,) = \begin{cases} w(z^+ - z^-)^2 & \text{if } u \le z^+ \le \bar{u},\ z^- = 0,\ y = 1 \\ w(z^+ - z^-)^2 & \text{if } l \le z^- \le \bar{l},\ \ z^+ = 0,\ y = 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$h(\,z^+\,,\,z^-\,,\,y\,) = w\left(\frac{(z^+)^2}{y} + \frac{(z^-)^2}{1-y}\right)$$

for $uy \le z^+ \le \bar{u}y$, $l(1-y) \le z^- \le \bar{l}(1-y)$, $y \in [0,1]$

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for $uy \leq z^+ \leq \bar{u}y$, $l(1-y) \leq z^- \leq \bar{l}(1-y)$, $y \in [0,1]$

- Can perform the projection analysis

$$g(z^+, z^-) = \min_{y \in [0,1]} \quad w\left( \frac{(z^+)^2}{y} + \frac{(z^-)^2}{1-y} \right)$$
$$uy \leq z^+ \leq \bar{u}y \;\;,\;\; l(1-y) \leq z^- \leq \bar{l}(1-y)$$

# The projected function

| cond. | $g(z^+, z^-)$ |
|---|---|
| $\bar{u} \leq l$ | $\begin{cases} \bar{u}((z^-)^2/(\bar{u} - z^+) + z^+) & \text{if } lz^+ + \bar{u}z^- \geq \bar{u}l \\ l((z^+)^2/(l - z^-) + z^-) & \text{if } lz^+ + \bar{u}z^- \leq \bar{u}l \end{cases}$ |
| $\bar{l} \leq u$ | $\begin{cases} u((z^-)^2/(u - z^+) + z^+) & \text{if } \bar{l}z^+ + uz^- \leq \bar{l}u \\ \bar{l}((z^+)^2/(\bar{l} - z^-) + z^-) & \text{if } \bar{l}z^+ + uz^- \geq \bar{l}u \end{cases}$ |
| $l \leq u \leq \bar{u}$ | $\begin{cases} u((z^-)^2/(u - z^+) + z^+) & \text{if } l \leq z^+ + z^- \leq u \\ (z^+ + z^-)^2 & \text{if } u \leq z^+ + z^- \leq \bar{u} \\ \bar{u}((z^-)^2/(\bar{u} - z^+) + z^+) & \text{if } \bar{u} \leq z^+ + z^- \leq \bar{l} \end{cases}$ |
| $l \leq u \leq \bar{l} \leq \bar{u}$ | $\begin{cases} u((z^-)^2/(u - z^+) + z^+) & \text{if } l \leq z^+ + z^- \leq u \\ (z^+ + z^-)^2 & \text{if } u \leq z^+ + z^- \leq \bar{l} \\ \bar{l}((z^+)^2/(\bar{l} - z^-) + z^-) & \text{if } \bar{l} \leq z^+ + z^- \leq \bar{u} \end{cases}$ |
| $u \leq l \leq \bar{l} \leq \bar{u}$ | $\begin{cases} l((z^+)^2/(l - z^-) + z^-) & \text{if } u \leq z^+ + z^- \leq l \\ (z^+ + z^-)^2 & \text{if } l \leq z^+ + z^- \leq \bar{l} \\ \bar{l}((z^+)^2/(\bar{l} - z^-) + z^-) & \text{if } \bar{l} \leq z^+ + z^- \leq \bar{u} \end{cases}$ |
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# The projected function: wrap-up

- $g(z^+, z^-)$ is piecewise-SOCP-quadratic with at most three pieces
- Under reasonable hypotheses, the "central" one is $(z^+ + z^-)^2$
- Symmetric bounds ($\bar{u}_i = \bar{l}_i$, $l_i = u_i$) $\Longrightarrow g(z^+, z^-) = (z^+ + z^-)^2$
- $w_i(z_i^+ - z_i^-)^2 \Longrightarrow w_i\left(z_i^+ + z_i^-\right)^2$ applicable also to nonsensitive cells
- Four possible "basic" formulations:

| MIQP | MIQP+ |
|---|---|
| $\min \sum_{i \in \mathcal{N}} w_i\left( (z_i^+)^2 + (z_i^-)^2 \right)$ | $\min \sum_{i \in \mathcal{N}} w_i\left( z_i^+ + z_i^- \right)^2$ |
| PR | PR+ |
| $\min \sum_{i \in \mathcal{U}} w_i\left((z_i^+)^2 + (z_i^-)^2\right) + \sum_{i \in \mathcal{S}} w_i\left(\frac{(z_i^+)^2}{y_i} + \frac{(z_i^-)^2}{1-y_i}\right)$ | $\min \sum_{i \in \mathcal{U}} w_i(z_i^+ + z_i^-)^2 + \sum_{i \in \mathcal{S}} w_i\left(\frac{(z_i^+)^2}{y_i} + \frac{(z_i^-)^2}{1-y_i}\right)$ |

- PR and PR+ solved with P/C or CP: P/C, CP, P/C+, CP+

# Computational experiments: symmetric instances

- Random one-hierarchical-two-dimensional data (1H2D), realistic size
- r-c-s(-a): rows-columns-%sensitive(-asymmetry $u_i = a \cdot l_i$), 10 each
- CP and CP+ always not competitive

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- CP and CP+ always not competitive

| | MIQP+ | | | P/C+ | | | MIQP | | | P/C | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | gap | time | nodes | gap | time | nodes | gap | time | nodes | gap | time | nodes |
| 10-20-3 | 0.01 | 442 | 474 | 0.00 | 486 | 357 | 6.49 | 9686 | 10365 | 0.00 | 1331 | 1973 |
| 10-20-5 | 0.01 | 765 | 690 | 0.01 | 1016 | 611 | 67.62 | 10000 | 2649 | 0.16 | 6695 | 8675 |
| 10-20-10 | 0.01 | 3852 | 10507 | 2.21 | 7660 | 2676 | 72.75 | 10000 | 5536 | 12.39 | 10000 | 3230 |
| 10-30-3 | 0.01 | 1470 | 760 | 0.01 | 1749 | 457 | 127.03 | 10000 | 778 | 0.98 | 9070 | 3022 |
| 10-30-5 | 0.01 | 4850 | 4003 | 0.07 | 7102 | 4769 | 118.53 | 10000 | 1422 | 15.80 | 10000 | 1853 |
| 10-30-10 | 2.44 | 10000 | 3512 | 8.26 | 10000 | 889 | 128.67 | 10000 | 1619 | 35.30 | 10000 | 643 |
| 20-20-3 | 0.00 | 1710 | 260 | 0.00 | 1874 | 291 | 158.64 | 10000 | 636 | 17.84 | 8559 | 596 |
| 20-20-5 | 0.01 | 3543 | 1507 | 1.27 | 7237 | 1185 | 138.59 | 10000 | 625 | 12.33 | 8808 | 481 |
| 20-20-10 | 7.10 | 10000 | 1968 | 24.51 | 10000 | 504 | 142.82 | 10000 | 777 | 38.22 | 10000 | 262 |
| 20-30-3 | 0.40 | 6113 | 738 | 3.60 | 6800 | 458 | 138.85 | 10000 | 726 | 27.17 | 10000 | 379 |
| 20-30-5 | 7.39 | 8791 | 751 | 15.19 | 8885 | 379 | 156.73 | 10000 | 801 | 32.83 | 10000 | 406 |
| 20-30-10 | 19.92 | 10000 | 674 | 32.04 | 10000 | 102 | 153.79 | 10000 | 496 | 44.06 | 10000 | 56 |

# Computational experiments: asymmetric instances 1

| | MIQP+ | | | P/C+ | | | MIQP | | | P/C | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | gap | time | nodes | gap | time | nodes | gap | time | nodes | gap | time | nodes |
| 10-20-3-2 | 0.00 | 23 | 9 | 0.00 | 58 | 1 | 0.01 | 1218 | 7823 | 0.00 | 106 | 17 |
| 10-20-3-5 | 0.01 | 19 | 1 | 0.00 | 82 | 1 | 0.01 | 322 | 197 | 0.00 | 111 | 1 |
| 10-20-3-10 | 0.00 | 15 | 7 | 0.00 | 55 | 1 | 0.01 | 270 | 124 | 0.00 | 78 | 1 |
| 10-20-5-2 | 0.01 | 58 | 30 | 0.00 | 119 | 9 | 0.04 | 10000 | 113601 | 0.00 | 152 | 32 |
| 10-20-5-5 | 0.01 | 21 | 15 | 0.00 | 79 | 1 | 0.01 | 1293 | 2332 | 0.00 | 81 | 1 |
| 10-20-5-10 | 0.00 | 20 | 2 | 0.00 | 106 | 1 | 0.01 | 1483 | 660 | 0.00 | 111 | 1 |
| 10-20-10-2 | 0.01 | 438 | 556 | 0.00 | 637 | 181 | 0.04 | 10000 | 67541 | 1.49 | 2904 | 370 |
| 10-20-10-5 | 0.01 | 4315 | 31344 | 0.00 | 142 | 1 | 0.08 | 10000 | 102641 | 0.00 | 142 | 1 |
| 10-20-10-10 | 0.01 | 416 | 2135 | 0.00 | 120 | 1 | 0.04 | 5044 | 26508 | 0.00 | 109 | 1 |
| 10-30-3-2 | 0.00 | 115 | 28 | 0.00 | 271 | 5 | 0.02 | 10000 | 55266 | 0.00 | 391 | 35 |
| 10-30-3-5 | 0.00 | 40 | 4 | 0.00 | 220 | 1 | 0.01 | 2447 | 1333 | 0.00 | 237 | 1 |
| 10-30-3-10 | 0.00 | 31 | 1 | 0.00 | 232 | 1 | 0.01 | 1468 | 565 | 0.00 | 258 | 1 |
| 10-30-5-2 | 0.00 | 193 | 103 | 0.00 | 377 | 19 | 0.05 | 10000 | 28721 | 0.00 | 455 | 72 |
| 10-30-5-5 | 0.01 | 119 | 39 | 0.00 | 333 | 1 | 0.01 | 4055 | 24181 | 0.00 | 258 | 1 |
| 10-30-5-10 | 0.01 | 63 | 46 | 0.00 | 207 | 1 | 0.01 | 1855 | 1104 | 0.00 | 216 | 1 |
| 10-30-10-2 | 0.01 | 1158 | 1035 | 0.00 | 1905 | 230 | 7.03 | 10000 | 27461 | 0.82 | 3066 | 986 |
| 10-30-10-5 | 0.01 | 6489 | 38818 | 0.00 | 401 | 1 | 8.53 | 10000 | 60347 | 0.00 | 311 | 1 |
| 10-30-10-10 | 0.01 | 4806 | 22519 | 0.00 | 522 | 1 | 0.09 | 10000 | 52141 | 0.00 | 372 | 1 |

- Easier than (not really) symmetric ones (under practitioners' control)

# Computational experiments: asymmetric instances 2

| | MIQP+ | | | P/C+ | | | MIQP | | | P/C | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | gap | time | nodes | gap | time | nodes | gap | time | nodes | gap | time | nodes |
| 20-20-3-2 | 0.00 | 136 | 25 | 0.00 | 393 | 1 | 0.03 | 10000 | 13721 | 0.00 | 502 | 9 |
| 20-20-3-5 | 0.01 | 72 | 1 | 0.00 | 625 | 1 | 0.01 | 4074 | 1207 | 0.00 | 691 | 1 |
| 20-20-3-10 | 0.00 | 76 | 1 | 0.00 | 574 | 1 | 2.18 | 5356 | 465 | 0.00 | 644 | 1 |
| 20-20-5-2 | 0.00 | 257 | 47 | 0.00 | 601 | 4 | 1.40 | 10000 | 14362 | 0.00 | 598 | 24 |
| 20-20-5-5 | 0.01 | 117 | 10 | 0.00 | 690 | 1 | 1.19 | 10000 | 15635 | 0.00 | 638 | 1 |
| 20-20-5-10 | 0.01 | 128 | 54 | 0.00 | 736 | 1 | 0.52 | 6434 | 2076 | 0.00 | 623 | 1 |
| 20-20-10-2 | 0.01 | 1448 | 212 | 0.00 | 2802 | 138 | 63.41 | 10000 | 1006 | 0.00 | 2525 | 228 |
| 20-20-10-5 | 0.02 | 9203 | 22462 | 0.00 | 943 | 1 | 3.40 | 10000 | 9950 | 0.00 | 634 | 1 |
| 20-20-10-10 | 0.03 | 7910 | 19421 | 0.00 | 1327 | 1 | 7.33 | 10000 | 9801 | 0.00 | 801 | 1 |
| 20-30-3-2 | 0.01 | 439 | 28 | 0.00 | 1477 | 1 | 13.94 | 10000 | 1203 | 0.00 | 1649 | 16 |
| 20-30-3-5 | 0.01 | 140 | 1 | 0.00 | 1597 | 1 | 5.39 | 8400 | 1767 | 0.00 | 1510 | 1 |
| 20-30-3-10 | 0.00 | 157 | 8 | 0.00 | 1601 | 1 | 8.34 | 9321 | 691 | 0.00 | 1547 | 1 |
| 20-30-5-2 | 0.00 | 777 | 65 | 0.00 | 2160 | 17 | 48.34 | 10000 | 612 | 0.00 | 2111 | 34 |
| 20-30-5-5 | 0.01 | 618 | 462 | 0.00 | 1800 | 1 | 19.74 | 10000 | 1692 | 0.00 | 1622 | 1 |
| 20-30-5-10 | 0.01 | 622 | 243 | 0.00 | 1988 | 1 | 2.14 | 9815 | 2623 | 0.00 | 1625 | 1 |
| 20-30-10-2 | 1.23 | 7575 | 1454 | 3.67 | 8407 | 297 | 79.80 | 10000 | 422 | 4.16 | 7705 | 262 |
| 20-30-10-5 | 0.52 | 10000 | 12890 | 0.00 | 2784 | 1 | 36.91 | 10000 | 718 | 0.00 | 1915 | 1 |
| 20-30-10-10 | 0.04 | 10000 | 17526 | 0.00 | 2619 | 1 | 27.08 | 10000 | 1441 | 0.00 | 1817 | 1 |

- MIQP+ better except for many sensitive cells & very asymmetric

# Approximated $P^2R$: Project&Lift

- Approximating the PR may work
- Biggest roadblocks in $P^2R$:
  - separability of $u$ (UC does not have it)
  - need for a special structure to be exploited (MV does not have it)
  - no $u \Longrightarrow$ home-made branch, cut, fixing, ...
- But for non-separable $u$, computing the projection is too complicated

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  - no $u \Longrightarrow$ home-made branch, cut, fixing, ...
- But for non-separable $u$, computing the projection is too complicated
- Possible solution: project as if $u$ were separable, re-introduce them
- Meanwhile, a few useful generalizations:
  - Extend to nonquadratic $f$ provided that for fixed $g^{\pm}$

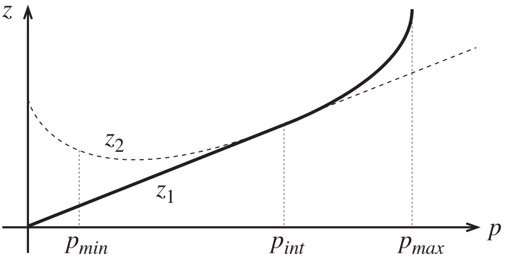
$$\tilde{u}(p) = \begin{cases} pg^{+} & \text{if } p \geq 0 \\ -pg^{-} & \text{if } p \leq 0 \end{cases} \tag{18}$$

    is the unique stationary point of $h(p, u)$ with respect to $u$
    (many cases: $p^k$, $e^p$, Kleinrock delay function, ...)
  - Extend to $\bar{p}_{min} < 0$ (4-pieces $z$ instead of 2-pieces one)

# Project&Lift

- The general form ($\bar{p}_{min} \geq 0$): for $p_{int} \in \{ \; \bar{p}_{min} \; , \; 1/g^+ \; , \; \bar{p}_{max} \; \}$

$$z(p) = \begin{cases} z_1(p) = \left( \; b + f(p_{int})/p_{int} + c/p_{int} \; \right) p & 0 \leq p \leq p_{int} \\ z_2(p) = f(p) + bp + c & p_{int} \leq p \leq \bar{p}_{max} \end{cases}$$



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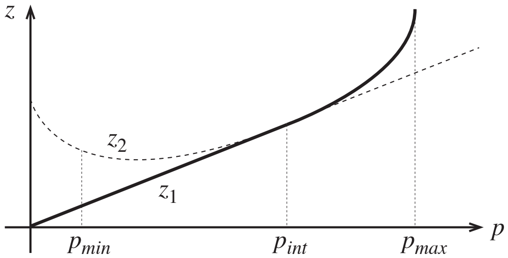
$$z(p) = \begin{cases} z_1(p) = \left(\ b + f(p_{int})/p_{int} + c/p_{int}\ \right)p & 0 \leq p \leq p_{int} \\ z_2(p) = f(p) + bp + c & p_{int} \leq p \leq \bar{p}_{max} \end{cases}$$




- Theorem: can be written as a NLP with the $u$ "making" $z_1$:

$$z(p) = \begin{cases} \min\ h(u,q) = & uz(p_{int}) + z_2(q + p_{int}) - z(p_{int}) \\ & (\bar{p}_{min} - p_{int})u \leq q \leq (\bar{p}_{max} - p_{int})u \\ & p = p_{int}u + q \quad , \quad u \in [0,1] \end{cases}$$

# Approximated Projected Perspective Reformulation

- With the integrality constraints, a reformulation of the block

$$
\begin{aligned}
\min \quad & u z(p_{int}) + z_2(q + p_{int}) - z(p_{int}) \\
& (\bar{p}_{min} - p_{int})u \leq q \leq (\bar{p}_{max} - p_{int})u \\
& p = p_{int} u + q \qquad , \qquad u \in \{0, 1\}
\end{aligned}
$$

  just use this instead of the original constraints

- Not as strong as PR, but same number of variables and easy to do

- 4-pieces version for $\bar{p}_{min} < 0$ (duplicate $u$)

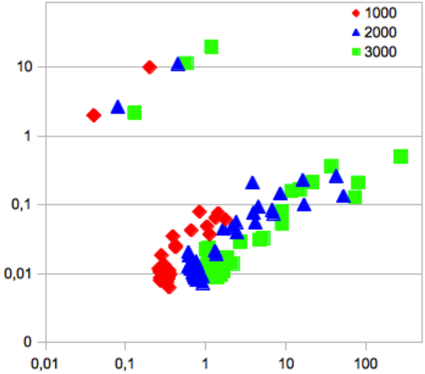
- Just properly translating one variable improves the LB:
  blatant violation of the no-free-lunch principle!

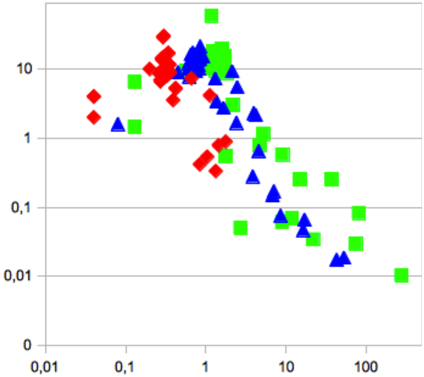
- Funny observation: $f(p) = ap^2$, just redefine $p = p_{int} u + q$ and use

$$
u^2 = u \qquad\qquad qu = q
$$

  valid because $u \in \{0, 1\}$ and strengthening (RLT?)

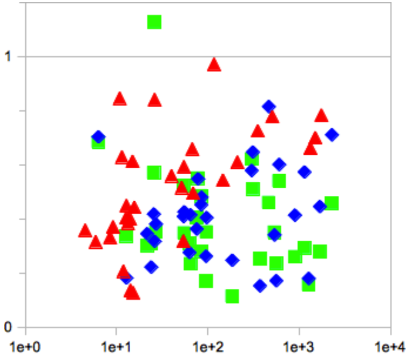
# Computational results: ND

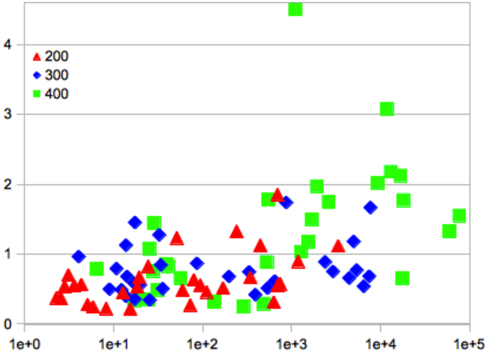



- left: ($AP^2R$ time) / (P/C time) plotted against $AP^2R$ time
- right = ($AP^2R$ time) / ($P^2R$ time) plotted against $AP^2R$ time

# Computational results: MV




- ($AP^2R$ time) / (P/C time) plotted against $AP^2R$ time
- right = (tight) constraint on max assets (10), not separable

# Computational results: UC

| | | AP$^2$R | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | NCNH | | CNH | | CH | | B&C | | | |
| $t$ | $h$ | time | dgap | time | dgap | time | pgap | nodes | time | pgap | gap |
| 10 | 0 | 0.31 | 1.49 | 1.50 | 0.29 | 2.59 | 0.04 | 1229 | 284 | 0.00 | 0.01 |
| 20 | 0 | 0.75 | 1.25 | 6.97 | 0.29 | 13.99 | 0.15 | 4635 | 9999 | 0.01 | 0.17 |
| 50 | 0 | 3.19 | 1.19 | 50.53 | 0.22 | 65.12 | 0.23 | 1078 | 9999 | 0.02 | 0.23 |
| 20 | 10 | 0.88 | 0.58 | 3.04 | 0.15 | 7.91 | 0.16 | 16477 | 1078 | 0.00 | 0.01 |
| 50 | 20 | 2.91 | 0.58 | 18.97 | 0.09 | 28.33 | | 3780 | 9999 | 0.00 | 0.07 |
| 75 | 35 | 5.46 | 0.49 | 39.18 | 0.06 | 45.73 | | 1727 | 9999 | 0.03 | 0.08 |
| | | PC | | | | | | | | | |
| 10 | 0 | 0.17 | 1.48 | 0.99 | 0.23 | 1.25 | 0.40 | 365 | 17 | 0.00 | 0.01 |
| 20 | 0 | 0.49 | 1.24 | 3.93 | 0.25 | 5.38 | | 15607 | 4851 | 0.00 | 0.02 |
| 50 | 0 | 2.85 | 1.16 | 16.59 | 0.19 | 20.63 | | 14286 | 9986 | 0.00 | 0.13 |
| 20 | 10 | 0.52 | 0.56 | 1.92 | 0.13 | 3.14 | 0.51 | 8107 | 240 | 0.00 | 0.01 |
| 50 | 20 | 2.05 | 0.57 | 6.17 | 0.07 | 13.11 | | 66945 | 6649 | 0.00 | 0.02 |
| 75 | 35 | 4.19 | 0.48 | 11.23 | 0.05 | 20.22 | 0.08 | 57456 | 9999 | 0.00 | 0.02 |

- No Cuts No Heuristic, Cuts No Heuristic, Cuts & Heuristic, B&C
- Sometimes a better root node solution is found, all the rest is bad

# Conclusions

- Errors can be useful, be glad for pesky referees/fellow researchers :-)

[7] Stubbs, Mehrotra "A branch-and-cut method for 0-1 mixed convex programming" *Math. Prog.*, 1999

[8] Khajavirad, Sahinidis "Convex envelopes generated from finitely many compact convex sets" *Math. Prog.* 2013

[9] Luedtke, Namazifar, Linderoth "Some results on the strength of relaxations of multilinear functions" *TR UW-Madison*, 2010

[10] F., Gentile, Lacalandra "Sequential Lagrangian-MILP Approaches for Unit Commitment Problems" *International Journal of Electrical Power and Energy Systems* 2011

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- MINLP a funny place to be in: nonlinear + combinatorial = many useful structures

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