Nonlinear Aspects of Routing in Telecommunication Networks

(MINLP meets computer networks)

A. Frangioni

with L. Galli M.G. Scutellà G. Stea

Dipartimento di Informatica, Università di Pisa

Dipartimento di Ingegneria dell'Informazione, Università di Pisa

INTERNATIONAL SCHOOL ON MATHEMATICS
"GUIDO STAMPACCHIA"
on Graph Theory, Algorithms and Applications (3rd Edition)
Erice, September 8–16, 2014

Outline

It Is Possible to Succumb to One's Success

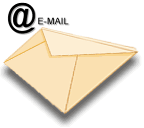
- The Internet was built around a set of assumptions:
  - Integrity of information is crucial: lost packets are retransmitted
  - Timeliness does not matter: the sooner the better, but no deadline
  - Application adapt to the available rate
    (higher rate $\Longleftrightarrow$ higher user satisfaction, but no QoS agreements)

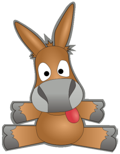
It Is Possible to Succumb to One's Success

- The Internet was built around a set of assumptions:
  - Integrity of information is crucial: lost packets are retransmitted
  - Timeliness does not matter: the sooner the better, but no deadline
  - Application adapt to the available rate
    (higher rate $\Longleftrightarrow$ higher user satisfaction, but no QoS agreements)
- $\Longrightarrow$ Packets don't count, can be: delayed (arbitrarily long), dropped, duplicated, displaced ($N+1$ arrives before N)

It Is Possible to Succumb to One's Success

- The Internet was built around a set of assumptions:
  - Integrity of information is crucial: lost packets are retransmitted
  - Timeliness does not matter: the sooner the better, but no deadline
  - Application adapt to the available rate
    (higher rate $\Longleftrightarrow$ higher user satisfaction, but no QoS agreements)
- $\Longrightarrow$ Packets don't count, can be: delayed (arbitrarily long), dropped, duplicated, displaced ($N+1$ arrives before N)
- $\Longrightarrow$ Internet is built upon the "Best Effort" Service Model: routers do their best to relay packets to destination, but no guarantee that a given packet will arrive at all
- Traditional Internet applications play by these rules



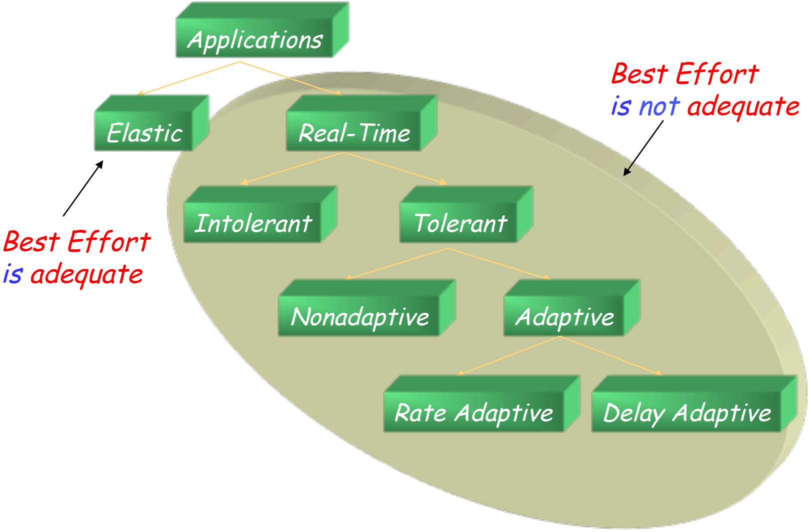




Succumbing to One's Success (cont.d)

- Despite this, Internet has became a huge splash hit (doh!)
- This has made some technologies (TCP-IP, Ethernet) dominant, economy of scale dictates convergence of everything:
  - traditional internet applications (+ social stuff)
  - IP Telephony
  - live Internet Protocol Television
  - online gaming/MMORPGs
  - industrial control systems
  - remote sensing and surveillance systems
  - M2M communication, IoT/IoE (pick your favorite buzzword)

  irrespectively of the access medium (fixed, cellular, WiFi, BLE, . . .)
- Clear issue: many of these completely unsuitable for best effort

Succumbing to One's Success (cont.d)



How to Avoid Succumbing to One's Success

- Now what? Introduce QoS guarantees
- What is QoS? "The ability of a network to offer different levels of service, in order to support different types of applications"

How to Avoid Succumbing to One's Success

- Now what? Introduce QoS guarantees
- What is QoS? "The ability of a network to offer different levels of service, in order to support different types of applications"

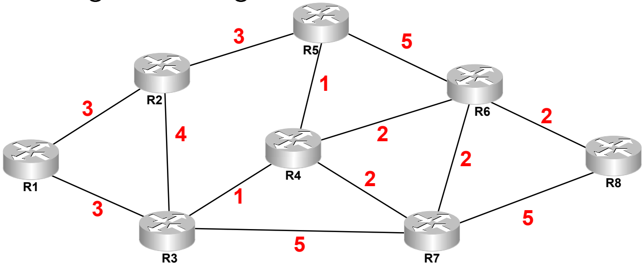
- Prime example: controlled end-to-end delay
- Critical in embedded systems (automative, avionics, ...)
- Much easier said than done, the provisions simply weren't there
- Introducing QoS is a complex, multi-faceted effort

Introducing QoS

- Requires adding ad hoc algorithms, hw/sw components, protocols:
  - simple, scalable and cost-effective (10^6 routers, 10^9 devices)
  - effective $\equiv$ guarantee that QoS objectives are met (money involved)
  - distributed and cooperating (no central control & management)
- Some building blocks have been designed, a few standardized
- Big issue: cooperation at the various timescales (vertical)
  - years/months: network design/expansion
  - weeks/days: resource provisioning (traffic engineering, routing)
  - hours/seconds: flow lifetime (resource reservation, admission control)
  - sub-millisecond: transmission (packet scheduling)
- Horizontal cooperation is also needed
- All this within a distributed decision model

QoS Requires Optimization (doh!)

- Example: setting OSPF weights in a domain

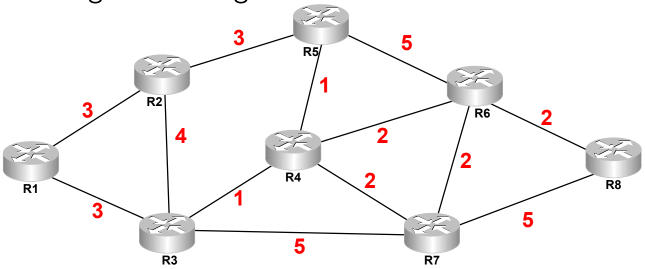


  ...a heinously complex problem for wanting too simple a system

- Select the "best" path for a flow (can be many, horrible in practice)

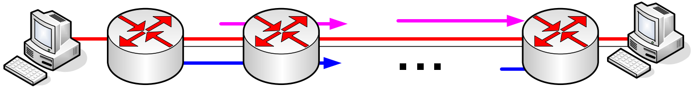
QoS Requires Optimization (doh!)

- Example: setting OSPF weights in a domain



...a heinously complex problem for wanting too simple a system

- Select the "best" path for a flow (can be many, horrible in practice)
- Packets, not circuits: how will the packets behave?
- Can't say unless you reserve capacity for the flow ($\approx$ circuits)

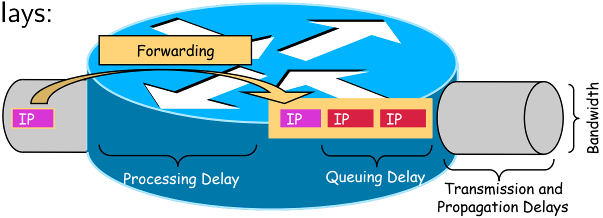


- How to do that optimally? It depends on many things

Outline

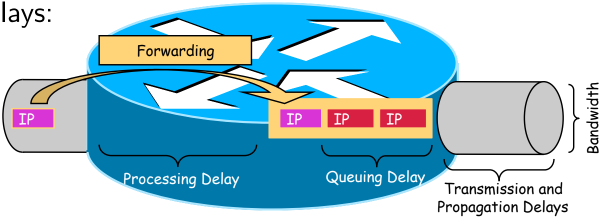
Flows, routers, links

- Flow: "distinguishable directed stream of packets with the same QoS requirements traveling from a source to one or more destinations"
- Lots of delays:



Flows, routers, links

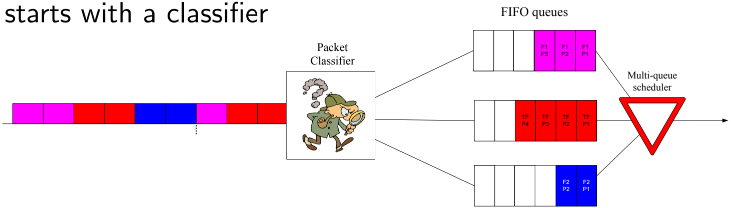
- Flow: "distinguishable directed stream of packets with the same QoS requirements traveling from a source to one or more destinations"
- Lots of delays:



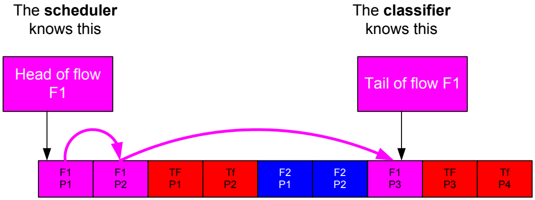
- Slowly creeping closer to our mathspeak:
  - IP Network $\equiv$ directed graph $G = (N, A)$ $(n = |N|,\ m = |A|)$
  - set of flows K: origin/destination (s^k, d^k), arrival curve $\mathcal{A}^k$ (???)
  - packet transmission cannot be preempted, for packets size matters: maximum transfer unit L (MTU, max. packet length)
  - $(i, j) \in A$: link speed (bandwidth) $w_{ij} \Longrightarrow$ link delay l_{ij} $(\geq L/w_{ij})$
  - $i \in N$: node processing delay n_i, assumed constant (!)
- Queuing delay a relevant factor, depends on packet schedulers

A (very brief) Intro To Packet Schedulers

- It all starts with a classifier



- Multiple logical lists in a single memory buffer space



- The crucial part is the scheduler

The Ideal Packet Scheduler

- What we would want from a packet scheduler:
  - simplicity (low complexity)
  - isolation of flows
  - controllability (parameters to alter the behavior)
  - fairness
  - guarantees

The Ideal Packet Scheduler

- What we would want from a packet scheduler:
  - simplicity (low complexity)
  - isolation of flows
  - controllability (parameters to alter the behavior)
  - fairness
  - guarantees
- Not at all easy
- Example: FIFO scheduler
  - simple: $O(1)$ ✓
  - no isolation of flows: a burst of a new flow can starve yours forever X
  - not controllable: can't change how it behaves X
  - no fairness: the first flow arriving takes it all X
  - no guarantees: can't prove anything on anything (e.g. max delay) X
- Strict priority list not much better

The Ideal Packet Scheduler: Generalized Processor Sharing

- What is the perfect formula of a scheduler?

The Ideal Packet Scheduler: Generalized Processor Sharing

- What is the perfect formula of a scheduler?
- Control: reserved rate r_{ij}^k such that $\bar{r}_{ij} = \sum_{k \in K} r_{ij}^k \leq w_{ij}$
- Schedule packets so that flow k achieves effective rate

$$r_{ij}^{eff,k} = \big(w_{ij} / \bar{r}_{ij} \big) r_{ij}^k \geq r_{ij}^k \qquad \equiv \qquad \text{delay} \; = L / r_{ij}^{eff,k}$$

$\equiv r_{ij}^k$ if the arc loaded, more if spare bandwidth available

- Provable perfect fairness (with appropriate definition)

The Ideal Packet Scheduler: Generalized Processor Sharing

- What is the perfect formula of a scheduler?
- Control: reserved rate r_{ij}^k such that $\bar{r}_{ij} = \sum_{k \in K} r_{ij}^k \leq w_{ij}$
- Schedule packets so that flow k achieves effective rate

$$r_{ij}^{eff,k} = (\, w_{ij} / \bar{r}_{ij} \,) r_{ij}^k \geq r_{ij}^k \qquad \equiv \qquad \text{delay} \; = L / r_{ij}^{eff,k}$$

$\equiv r_{ij}^k$ if the arc loaded, more if spare bandwidth available

- Provable perfect fairness (with appropriate definition)
- Can this be achieved? Almost, but not quite
- For once, GPS defined for idealized fluid model but we have packets
- Furthermore, it cannot be done in less than $O(\log |K|)$ (no $O(1)$)
- Yet, $O(\log |K|)$ good approximations exist
  (e.g. Worst-case Fair Weighted Fair Queuing — WF^{2}Q)

Good Approximations To The Ideal Packet Scheduler

- Notation: $r^k_{ij} > 0 \Longrightarrow$ flow k passes through $(i,j) \Longrightarrow k \in P(i,j)$
  $r^{min}_{ij} = \min\{\, r^k_{ij} \,:\, k \in P(i,j)\,\}$
- Actual scheduling protocols (others $\exists$, e.g. group-based SRP approx.)

$$\theta^k_{ij} = \frac{L}{w_{ij}} + \begin{cases} L/r^{eff,k}_{ij} & \text{if } P(i,j) \setminus \{k\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \begin{array}{c}\text{Strictly}\\ \text{Rate-Proportional}\end{array} \tag{1}$$

$$\theta^k_{ij} = \Big(|P(i,j)| + 1 \Big) \frac{L}{w_{ij}} + \frac{L}{r^{eff,k}_{ij}} \quad \begin{array}{c}\text{Weakly}\\ \text{Rate-Proportional}\end{array} \tag{2}$$

$$\theta^k_{ij} = \left(|P(i,j)| + \frac{\bar{r}_{ij}}{r^{min}_{ij}} \right) \frac{L}{w_{ij}} + \frac{L}{r^{eff,k}_{ij}} \quad \text{Frame-Based} \tag{3}$$

L/w_{ij}: a packet has to be entirely received before anything happens

Good Approximations To The Ideal Packet Scheduler

- Notation: $r_{ij}^{k} > 0 \Longrightarrow$ flow k passes through $(i,j) \Longrightarrow k \in P(i,j)$
  $r_{ij}^{min} = \min\{\, r_{ij}^{k} \,:\, k \in P(i,j)\,\}$
- Actual scheduling protocols (others $\exists$, e.g. group-based SRP approx.)

$$\theta_{ij}^{k} = \frac{L}{w_{ij}} + \begin{cases} L/r_{ij}^{eff,k} & \text{if } P(i,j) \setminus \{k\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \text{Strictly Rate-Proportional} \tag{1}$$

$$\theta_{ij}^{k} = \Big(|P(i,j)| + 1\Big)\frac{L}{w_{ij}} + \frac{L}{r_{ij}^{eff,k}} \qquad \text{Weakly Rate-Proportional} \tag{2}$$

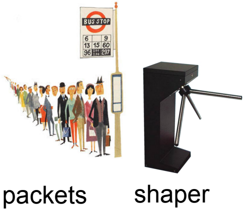
$$\theta_{ij}^{k} = \left(|P(i,j)| + \frac{\bar{r}_{ij}}{r_{ij}^{min}}\right)\frac{L}{w_{ij}} + \frac{L}{r_{ij}^{eff,k}} \qquad \text{Frame-Based} \tag{3}$$

L/w_{ij}: a packet has to be entirely received before anything happens

- SRP $\leq$ WRP $\leq$ FB
- SRP is $O(\log|K|)$, WRP is $O(\log|K|)$ but simpler, FB is $O(1)$

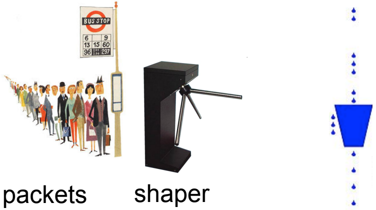
Putting It All Together

- Given all individual pieces, compute the end-to-end delay (e2ed)
- Could use queuing theory, but it would be very complex; plus: do your really know the arrival distribution?
- Alternative: worst case analysis, using network calculus
- Last crucial ingredient: the arrival function $\mathcal{A}^k$
- Not trivial to determine, but a nice trick: traffic shaper



Putting It All Together

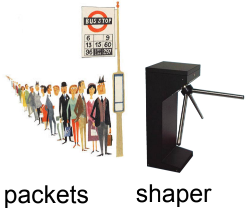
- Given all individual pieces, compute the end-to-end delay (e2ed)
- Could use queuing theory, but it would be very complex; plus: do your really know the arrival distribution?
- Alternative: worst case analysis, using network calculus
- Last crucial ingredient: the arrival function $\mathcal{A}^k$
- Not trivial to determine, but a nice trick: traffic shaper

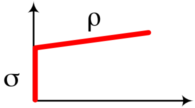


- In particular, leaky-bucket traffic shaper with burst σ^k and rate ρ^k

Putting It All Together

- Given all individual pieces, compute the end-to-end delay (e2ed)
- Could use queuing theory, but it would be very complex; plus: do your really know the arrival distribution?
- Alternative: worst case analysis, using network calculus
- Last crucial ingredient: the arrival function $\mathcal{A}^k$
- Not trivial to determine, but a nice trick: traffic shaper





- In particular, leaky-bucket traffic shaper with burst σ^k and rate ρ^k makes for a very simple arrival function

The Worst-case End-To-End Delay Formula (at last!)

- Worst-case e2ed (WCD) of flow k with σ^k, ρ^k depends on:
  1. the selected s^k–d^k path P^k in G;
  2. the reserved rates $r^k_{ij} \in (0, w_{ij}]$ for each $(i, j) \in P^k$
  3. the specific packet scheduler

The Worst-case End-To-End Delay Formula (at last!)

- Worst-case e2ed (WCD) of flow k with σ^k, ρ^k depends on:
  1. the selected s^k–d^k path P^k in G;
  2. the reserved rates $r^k_{ij} \in (0, w_{ij}]$ for each $(i,j) \in P^k$
  3. the specific packet scheduler
- Necessary assumption for finite WCD:
  $r^k_{ij} \geq \rho^k$ for each $(i,j) \in P^k \equiv r^k_{min} = \min\{\, r^k_{ij} \;:\; (i,j) \in P^k \,\} \geq \rho^k$
  (rate $\rho^k \equiv$ "steady-state" flow demand in usual flow models)

The Worst-case End-To-End Delay Formula (at last!)

- Worst-case e2ed (WCD) of flow k with σ^k, ρ^k depends on:
  1. the selected s^k–d^k path P^k in G;
  2. the reserved rates $r^k_{ij} \in (0, w_{ij}]$ for each $(i,j) \in P^k$
  3. the specific packet scheduler
- Necessary assumption for finite WCD:
  $r^k_{ij} \geq \rho^k$ for each $(i,j) \in P^k \equiv r^k_{min} = \min\{\, r^k_{ij} \;:\; (i,j) \in P^k \,\} \geq \rho^k$
  (rate $\rho^k \equiv$ "steady-state" flow demand in usual flow models)
- General WCD formula (nonlinear!):

$$\frac{\sigma^k}{r^k_{min}} + \sum_{(i,j)\in P^k} \left(\theta^k_{ij} + l_{ij} + n_i \right) \tag{4}$$

  where θ^k_{ij} is the protocol-specific arc delay (also nonlinear!)
- σ^k / r^k_{min}: the burst can happen just before the worst-case packet, all of it has to go through the bottleneck arc
- Good news: (4) convex and SOCP-representable if θ^k_{ij} is

Outline

Delay Constrained Routing Problems

Delay Constrained Routing Problem (DCR)

Compute paths and reserve resources on arcs at minimum cost such that the maximum delay of each flow is $\leq$ deadline

Delay Constrained Routing Problems

Delay Constrained Routing Problem (DCR)

Compute paths and reserve resources on arcs at minimum cost such that the maximum delay of each flow is $\leq$ deadline

- Single-Flow Single-Path (SFSP) DCR: one new unsplittable flow (just about to enter the network, has to be routed now)
  - drop superscripts, r^k_{ij} = existing flows, fixed
  - $P(i,j)$ = set of paths passing through (i,j) excluding the new one
  - $\bar{r}_{ij} = \sum_{k \in P(i,j)} r^k_{ij}$, $r^{min}_{ij} = \min\{\, r^k_{ij} \,:\, k \in P(i,j)\,\}$ exclude new flow
- Fixed deadline δ on the new flow
- Reservable capacity $w_{ij} \geq w_{ij} - \bar{r}_{ij} \geq c_{ij} \geq r_{ij}$
- Linear capacity reservation cost f_{ij} (often $= 1 \equiv$ Equal Cost (EC))
- Assumption: all the other flows must remain feasible (access control)

A (partial) MI-SOCP Model for SFSP-DCR

- Path binary variables x_{ij}, reserve continuous variables r_{ij}

$$\min \sum_{(i,j)\in A} f_{ij} r_{ij} \tag{5}$$

$$\sum_{(j,i)\in BS(i)} x_{ji} - \sum_{(i,j)\in FS(i)} x_{ij} = \begin{cases} -1 & \text{if } i = s \\ 1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases} \qquad i \in N \tag{6}$$

$$0 \leq r_{ij} \leq c_{ij} x_{ij} \qquad (i,j) \in A \tag{7}$$

$$\rho \leq r_{min} \leq r_{ij} + c_{max}(1 - x_{ij}) \qquad (i,j) \in A \tag{8}$$

$$t + \sum_{(i,j)\in A} \big(\theta_{ij} + (l_{ij} + n_i) x_{ij} \big) \leq \delta \tag{9}$$

$$t\, r_{min} \geq \sigma \quad , \quad t \geq 0 \tag{10}$$

$$x_{ij} \in \{0,1\} \quad , \quad r_{ij} \in \mathbb{R} \qquad (i,j) \in A$$

- (10) rotated SOCP constraint $\equiv t \geq \sigma / r_{min}$ (since $t \geq 0$)
- $c_{max} = \max\{ c_{ij} \;:\; (i,j) \in A \} =$ big-M, but cannot use c_{ij} (otherwise $r_{min} \leq c_{ij}$ even if $(i,j) \notin P$)

"Bound" Versions of the (Worst-case) Delay Formulæ

- Worst-worst case: $r_{ij}^{eff} = r_{ij}$, $P(i,j) \neq \emptyset \Longrightarrow$ coarser (but valid) estimate of the delay, somewhat simplified formulæ:

$$\theta_{ij} = \frac{L}{r_{ij}} + \frac{L}{w_{ij}} \qquad \text{SRP} \quad (11)$$

$$\theta_{ij} = \frac{L}{r_{ij}} + |P(i,j)| \frac{L}{w_{ij}} \qquad \text{WRP} \quad (12)$$

$$\theta_{ij} = \frac{L}{r_{ij}} + \left(|P(i,j)| + \frac{w_{ij} - r_{ij}}{\min\{\, r_{ij} \,,\, r_{ij}^{min} \,\}} \right) \frac{L}{w_{ij}} \qquad \text{FB} \quad (13)$$

- (11) independent of other flows, convex, SOCP-representable
- (12) $\approx$ (11) but not flow-independent
- (13) (surprisingly) also convex but only for SFSP, less trivial
- (12) and (13) not flow-independent $\Longrightarrow$ have admission control issue

A big-M Formulation for SRP-SFSP-DCR

- $\theta_{ij} = L / r_{ij}$, add L / w_{ij} to the coefficient of x_{ij} in (9)
- Issue: how to write "$x_{ij} = 1 \Longrightarrow \theta_{ij} = L / r_{ij}$, $x_{ij} = 0 \Longrightarrow \theta_{ij} = 0$"; can't use $r_{ij} \, \theta_{ij} \geq L$ for that $\Longrightarrow \theta_{ij} > 0$ always

A big-M Formulation for SRP-SFSP-DCR

- $\theta_{ij} = L / r_{ij}$, add L / w_{ij} to the coefficient of x_{ij} in (9)
- Issue: how to write "$x_{ij} = 1 \Longrightarrow \theta_{ij} = L / r_{ij}$, $x_{ij} = 0 \Longrightarrow \theta_{ij} = 0$"; can't use $r_{ij}\,\theta_{ij} \geq L$ for that $\Longrightarrow \theta_{ij} > 0$ always
- Solution: two extra sets of variables s_{ij} and r'_{ij}

$$0 \leq \theta_{ij} \leq M x_{ij}$$

$$\theta_{ij} \geq s_{ij} - M(1 - x_{ij})$$

$$s_{ij}\, r'_{ij} \geq L \quad , \quad s_{ij} \geq 0$$

$$0 \leq r'_{ij} \leq r_{ij} + M(1 - x_{ij})$$

  - $\theta_{ij} \geq s_{ij}$ if $x_{ij} = 1$, while θ_{ij} and s_{ij} are "free" if $x_{ij} = 0$
  - $r'_{ij} \leq r_{ij}$ if $x_{ij} = 1$, while r'_{ij} and r_{ij} are "free" if $x_{ij} = 0$
  - $s_{ij} \geq L / r'_{ij} \Longrightarrow \theta_{ij} \geq s_{ij} \geq L / r'_{ij} \geq L / r_{ij}$ if $x_{ij} = 1$
- $M = \max(\sqrt{L}\,,\ L / \rho)$ suffices, still it's big-M: can we do better?

Outline

A Step Up: Convex Indicator Constraints

- Vector of n variables x, convex function $f(x)$, one binary variable y
- Constraint $f(x) \leq 0$ "active" $\iff y = 1$, or more in general, constraint $\ldots + s + \ldots \leq d$ with $s = f(x)$ if $y = 1$, $s = 0$ otherwise

A Step Up: Convex Indicator Constraints

- Vector of n variables x, convex function $f(x)$, one binary variable y
- Constraint $f(x) \leq 0$ "active" $\iff$ $y = 1$, or more in general, constraint $\ldots + s + \ldots \leq d$ with $s = f(x)$ if $y = 1$, $s = 0$ otherwise
- Union of $\mathcal{P}_0 = \{\, (x, 0) \in \mathbb{R}^{n+1} \;:\; l_0 \leq x \leq u_0 \,\}$
  $\mathcal{P}_1 = \{\, (x, 1) \in \mathbb{R}^{n+1} \;:\; l_1 \leq x \leq u_1 \,,\, f(x) \leq 0 \,\}$
- Special case: $\mathcal{P}_0 = \{\, (0, 0) \,\}$, i.e., $l_0 = u_0 = 0$

A Step Up: Convex Indicator Constraints

- Vector of n variables x, convex function $f(x)$, one binary variable y
- Constraint $f(x) \leq 0$ "active" $\Longleftrightarrow$ $y = 1$, or more in general, constraint $\ldots + s + \ldots \leq d$ with $s = f(x)$ if $y = 1$, $s = 0$ otherwise
- Union of
$$\mathcal{P}_0 = \{ (x, 0) \in \mathbb{R}^{n+1} \ : \ l_0 \leq x \leq u_0 \}$$
$$\mathcal{P}_1 = \{ (x, 1) \in \mathbb{R}^{n+1} \ : \ l_1 \leq x \leq u_1 \,,\ f(x) \leq 0 \}$$
- Special case: $\mathcal{P}_0 = \{ (0, 0) \}$, i.e., $l_0 = u_0 = 0$
- Obvious MINLP formulations: $yl_1 \leq x \leq yu_1$ plus
$$f(y) \leq M(1 - y) \qquad \text{or} \qquad s \geq 0 \ , \ \ s \geq f(x) - M(1 - y)$$
- Continuous relaxation can be very weak: M "large"
- What can we do to improve on this? If f is linear, nothing . . .

A Step Up: Convex Indicator Constraints

- Vector of n variables x, convex function $f(x)$, one binary variable y
- Constraint $f(x) \leq 0$ "active" $\iff$ $y = 1$, or more in general, constraint $\ldots + s + \ldots \leq d$ with $s = f(x)$ if $y = 1$, $s = 0$ otherwise
- Union of $\mathcal{P}_0 = \{\, (x, 0) \in \mathbb{R}^{n+1} \; : \; l_0 \leq x \leq u_0 \,\}$
  $\mathcal{P}_1 = \{\, (x, 1) \in \mathbb{R}^{n+1} \; : \; l_1 \leq x \leq u_1 \, , \; f(x) \leq 0 \,\}$
- Special case: $\mathcal{P}_0 = \{\, (0, 0) \,\}$, i.e., $l_0 = u_0 = 0$
- Obvious MINLP formulations: $yl_1 \leq x \leq yu_1$ plus

$$f(y) \leq M(1 - y) \qquad \text{or} \qquad s \geq 0 \;\; , \;\; s \geq f(x) - M(1 - y)$$

- Continuous relaxation can be very weak: M "large"
- What can we do to improve on this? If f is linear, nothing . . .
  . . . but if f is nonlinear, we can indeed do something

Constructing a better formulation

- General result: $conv(\mathcal{P}_0 \cup \mathcal{P}_1) = pr_{(p,u)}(cl(\mathcal{P}^*))$, where

$$\mathcal{P}^* = \left\{ \begin{array}{l} (x, x', y) \in \mathbb{R}^{2n+1} \; : \; y\, f(x'/y) \leq 0 \; , \; y \in (0, 1] \\ y l_1 \leq x' \leq y u_1 \; , \; (1 - y) l_0 \leq x - x' \leq (1 - y) u_0 \end{array} \right.$$

  the best possible convex approximation of their (nonconvex) union

Constructing a better formulation

- General result: $conv(\mathcal{P}_0 \cup \mathcal{P}_1) = pr_{(p,u)}(cl(\mathcal{P}^*))$, where

$$\mathcal{P}^* = \left\{ \begin{array}{l} (x, x', y) \in \mathbb{R}^{2n+1} \;:\; y\, f(x'/y) \leq 0 \;,\; y \in (0,1] \\ y l_1 \leq x' \leq y u_1 \;,\; (1-y) l_0 \leq x - x' \leq (1-y) u_0 \end{array} \right.$$

  the best possible convex approximation of their (nonconvex) union

- Simplifies somewhat for $\mathcal{P}_0 = \{ (0,0) \}$ (and f "nice"):

  $$conv(\mathcal{P}_0 \cup \mathcal{P}_1) = \left\{ (x, y) \;:\; y l_1 \leq x \leq y u_1 \,,\; y\, f(x/y) \leq 0 \,,\; y \in [0,1] \right\}$$

Constructing a better formulation

- General result: $conv(\mathcal{P}_0 \cup \mathcal{P}_1) = pr_{(p,u)}(cl(\mathcal{P}^*))$, where

$$\mathcal{P}^* = \left\{ \begin{array}{l} (x, x', y) \in \mathbb{R}^{2n+1} \;:\; y\, f(x'/y) \leq 0\,,\; y \in (0,1] \\ y l_1 \leq x' \leq y u_1\,,\; (1-y) l_0 \leq x - x' \leq (1-y) u_0 \end{array} \right.$$

  the best possible convex approximation of their (nonconvex) union

- Simplifies somewhat for $\mathcal{P}_0 = \{\, (0,0) \,\}$ (and f "nice"):

  $$conv(\, \mathcal{P}_0 \cup \mathcal{P}_1 \,) = \Big\{\, (x, y) \;:\; y l_1 \leq x \leq y u_1\,,\; y\, f(x/y) \leq 0\,,\; y \in [0,1] \,\Big\}$$

- Even simpler to see: nonlinear convex-cost semi-continuous variable

$$f(x, y) = \begin{cases} 0 & \text{if } y = 0 \text{ and } x = 0 \\ f(x) + c & \text{if } y = 1 \text{ and } l_1 \leq x \leq u_1 \\ +\infty & \text{otherwise} \end{cases}$$

  whose convex envelope (assuming $0f(0/0) = 0$ and f nice) is

$$\overline{co} f(x, y) = \begin{cases} y f(x/y) + cy & \text{if } y l_1 \leq x \leq y u_1\,,\; y \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

Constructing a better formulation

- General result: $conv(\mathcal{P}_0 \cup \mathcal{P}_1) = pr_{(p,u)}(cl(\mathcal{P}^*))$, where

$$\mathcal{P}^* = \left\{ \begin{array}{l} (x, x', y) \in \mathbb{R}^{2n+1} \;:\; y\, f(x'/y) \leq 0 \;,\; y \in (0,1] \\ y l_1 \leq x' \leq y u_1 \;,\; (1-y) l_0 \leq x - x' \leq (1-y) u_0 \end{array} \right.$$

  the best possible convex approximation of their (nonconvex) union

- Simplifies somewhat for $\mathcal{P}_0 = \{\, (0,0) \,\}$ (and f "nice"):

  $$conv(\, \mathcal{P}_0 \cup \mathcal{P}_1 \,) = \Big\{\, (x,y) \;:\; y l_1 \leq x \leq y u_1 \,,\; y\, f(x/y) \leq 0 \,,\; y \in [0,1] \,\Big\}$$

- Even simpler to see: nonlinear convex-cost semi-continuous variable

$$f(x,y) = \begin{cases} 0 & \text{if } y = 0 \text{ and } x = 0 \\ f(x) + c & \text{if } y = 1 \text{ and } l_1 \leq x \leq u_1 \\ +\infty & \text{otherwise} \end{cases}$$

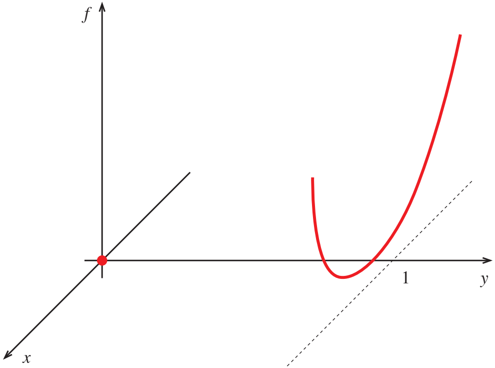
  whose convex envelope (assuming $0 f(0/0) = 0$ and f nice) is

$$\overline{co} f(x,y) = \begin{cases} y f(x/y) + cy & \text{if } y l_1 \leq x \leq y u_1 \;,\; y \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

- $f(x,y) = y\, f(x/y)$ is the perspective function of f

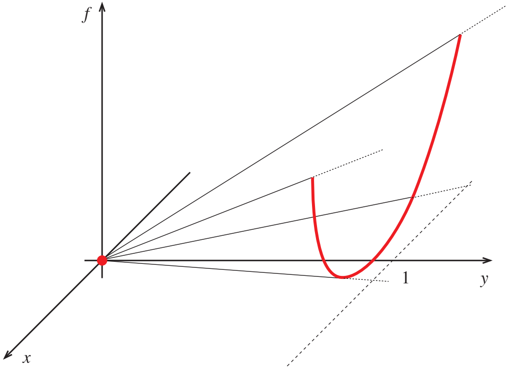
The Perspective what?

- $f(x, y) = y\,f(x/y)$ is convex for $y > 0$ if f is
- *epi* $f(x, y)$ is a cone emanating from $(0, 0)$ with the "shape of f"



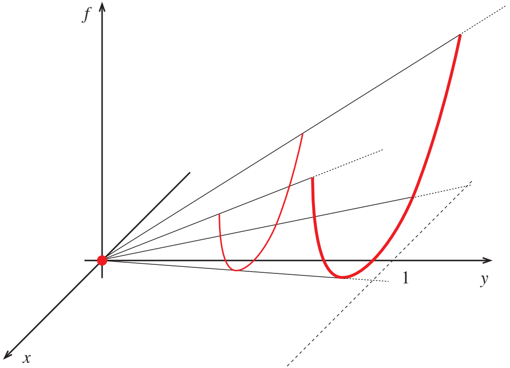
The Perspective what?

- $\tilde{f}(x, y) = y f(x/y)$ is convex for $y > 0$ if f is
- *epi* $\tilde{f}(x, y)$ is a cone emanating from $(0, 0)$ with the "shape of f"



The Perspective what?

- $\tilde{f}(x, y) = y\, f(x/y)$ is convex for $y > 0$ if f is
- *epi* $\tilde{f}(x, y)$ is a cone emanating from $(0, 0)$ with the "shape of f"

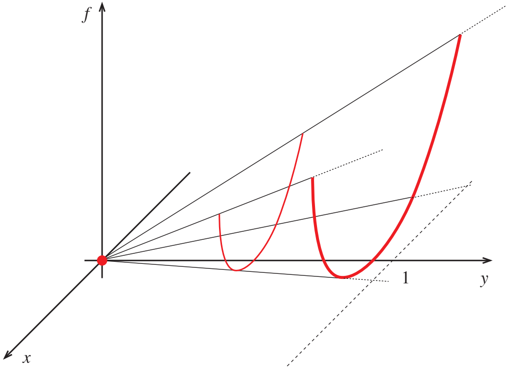


- $\tilde{f}(x, y)$ "much more nonlinear" than $f(x) + cy$

example: $f(x) = ax^2 + bx \quad \Longrightarrow \quad \tilde{f}(x, y) = (a/y)x^2 + bx + cy$

The Perspective what?

- $f(x,y) = y\,f(x/y)$ is convex for $y > 0$ if f is
- *epi* $f(x,y)$ is a cone emanating from $(0,0)$ with the "shape of f"



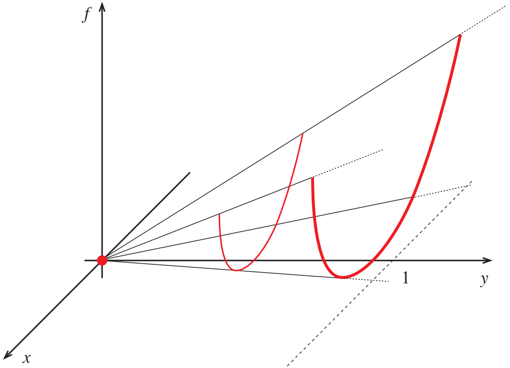
- $f(x,y)$ "much more nonlinear" than $f(x) + cy$

example: $f(x) = ax^2 + bx \quad \Longrightarrow \quad f(x,y) = (a/y)x^2 + bx + cy$

notes: I) $a/y > a$ for $y < 1$;

The Perspective what?

- $f(x, y) = y\,f(x/y)$ is convex for $y > 0$ if f is
- *epi* $f(x, y)$ is a cone emanating from $(0, 0)$ with the "shape of f"



- $f(x, y)$ "much more nonlinear" than $f(x) + cy$

example: $f(x) = ax^2 + bx \quad \Longrightarrow \quad f(x, y) = (a/y)x^2 + bx + cy$

notes: I) $a/y > a$ for $y < 1$; II) for $a = 0$ nothing happens

The Perspective Reformulation (Relaxation)

- Slightly more general: $Ax \leq b$ compact ($\equiv \{Ax \leq 0\} = \{0\}$), MINLP

$$\min \{ \; f(x) + cy \; : \; Ax \leq by \; , \; y \in \{0, 1\} \; \} \tag{14}$$

- Its continuous relaxation: convex, but weak bound

$$\min \{ \; f(x) + cy \; : \; Ax \leq by \; , \; y \in [0, 1] \; \} \tag{15}$$

The Perspective Reformulation (Relaxation)

- Slightly more general: $Ax \leq b$ compact ($\equiv \{Ax \leq 0\} = \{0\}$), MINLP

$$\min \{\ f(x) + cy \ :\ Ax \leq by\ ,\ y \in \{0,1\}\ \} \tag{14}$$

- Its continuous relaxation: convex, but weak bound

$$\min \{\ f(x) + cy \ :\ Ax \leq by\ ,\ y \in [0,1]\ \} \tag{15}$$

- Better relaxation (best possible convex one):

$$\min \{\ yf(x/y) + cy \ :\ Ax \leq by\ ,\ y \in [0,1]\ \} \tag{16}$$

better lower bound than (15), still convex, but "more nonlinear"

The Perspective Reformulation (Relaxation)

- Slightly more general: $Ax \leq b$ compact ($\equiv \{Ax \leq 0\} = \{0\}$), MINLP

$$\min \{\ f(x) + cy \ :\ Ax \leq by \ ,\ y \in \{0,1\}\ \} \tag{14}$$

- Its continuous relaxation: convex, but weak bound

$$\min \{\ f(x) + cy \ :\ Ax \leq by \ ,\ y \in [0,1]\ \} \tag{15}$$

- Better relaxation (best possible convex one):

$$\min \{\ yf(x/y) + cy \ :\ Ax \leq by \ ,\ y \in [0,1]\ \} \tag{16}$$

  better lower bound than (15), still convex, but "more nonlinear"

- Even better: (16) continuous relaxation of Perspective Reformulation

$$\min \{\ y\,f(x/y) + cy \ :\ Ax \leq by \ ,\ y \in \{0,1\}\ \} \tag{17}$$

  $\equiv$ (14) (requires assuming $0f(0/0) = 0$, not really an issue)

Solving the Perspective Relaxation I

- But how to solve (16) it efficiently?

Solving the Perspective Relaxation I

- But how to solve (16) it efficiently?
- Good news: $y\,f(x/y)$ is SOCP-representable if f is
- Example 1: for $f(x) = ax^2 + bx$, (17) becomes

$$\min \left\{ t + bx + cy \ : \ ax^2 \leq ty \, , \, Ax \leq by \, , \, y \in \{0,1\} \right\} \tag{18}$$

a Mixed-Integer (rotated) Second-Order Cone Program

Solving the Perspective Relaxation I

- But how to solve (16) it efficiently?
- Good news: $y\,f(x/y)$ is SOCP-representable if f is
- Example 1: for $f(x) = ax^2 + bx$, (17) becomes

$$\min\left\{\, t + bx + cy \;:\; ax^2 \leq ty\,,\; Ax \leq by\,,\; y \in \{0,1\}\,\right\} \tag{18}$$

  a Mixed-Integer (rotated) Second-Order Cone Program

- Example 2: $f(\theta, r) = L/r - \theta \leq 0 \Longleftrightarrow x = 1$ gives

$$Lx^2/r \leq \theta \quad\equiv\quad Lx^2 \leq \theta r$$

  if $x = 0$ then θ can be 0 whatever r, if $x = 1$ then $\theta \geq L/r$

- Note: Lx/r would be even better, but it is not convex; in fact, $L0/0 \neq 0$, whereas $L0^2/0 = 0$

Solving the Perspective Relaxation II

- Is it the only way? Of course not.

- Every convex function is the supremum of its affine minorants

  $(v, x, y) \in epi\ f \iff Ax \leq by,\ y \in [0, 1]$, and $\forall \bar{x}$ s.t. $A\bar{x} \leq b$

  $$v \geq f(\bar{x}) + c + \left[\, s \ , \ c + f(\bar{x}) - s\bar{x} \,\right] \left[\begin{array}{c} x - \bar{x} \\ y - 1 \end{array} \right] \qquad \forall s \in \partial f(\bar{x})$$

- Infinitely many inequalities (possibly "twice" if f nonsmooth at $\bar{x}$); looks though, but actually pretty OK for B&C (with some ε)

- The quadratic case: Perspective Cuts (P/C)

  $$v \geq (2a\bar{x} + b)x + (c - a\bar{x}^2)y \qquad \forall \bar{x} \text{ s.t. } A\bar{x} \leq b$$

- Basically the same thing as linearizing the cones in (18), which can be done automatically ... but does not work nearly as well (don't know why)

Application to SRP-SFSP-DCR

- "Perspectivized" formulation (first two not even strictly necessary):

$$\rho x_{ij} \leq r_{ij} \leq c_{ij} x_{ij} \quad , \quad 0 \leq \theta_{ij} \leq (L/\rho) x_{ij} \quad , \quad \theta_{ij} r_{ij} \geq L x_{ij}^2$$

  original variables + a(nother rotated) SOCP constraint

Application to SRP-SFSP-DCR

- "Perspectivized" formulation (first two not even strictly necessary):

$$\rho x_{ij} \leq r_{ij} \leq c_{ij} x_{ij} \quad , \quad 0 \leq \theta_{ij} \leq (L/\rho) x_{ij} \quad , \quad \theta_{ij} r_{ij} \geq L x_{ij}^2$$

  original variables + a(nother rotated) SOCP constraint

- Looks much better than

$$\begin{aligned} & 0 \leq \theta_{ij} \leq M x_{ij} \\ & \theta_{ij} \geq s_{ij} - M(1 - x_{ij}) \\ & s_{ij}\, r'_{ij} \geq L \ , \ \ s_{ij} \geq 0 \\ & 0 \leq r'_{ij} \leq r_{ij} + M(1 - x_{ij}) \end{aligned}$$

  not only better bound, but also fewer variables/constraints

- Is it? Time for computational tests

- Don't even bother with linearizations, just call a general-purpose MI-SOCP solver

Instances

- Real-world IP network topologies (`GARR`, `SNDlib`, `TopoZOO`): 10 – 65 nodes, 12 – 170 arcs, few 10s – several 100s flows
- Realistic random topologies (`Waxman` model): $\leq$ 200 nodes, 1500 arcs
- Equal (reservation) Costs $f_{ij} = 1$
- `FNSS` tool for realistic traffic matrices ($\mu(T) = 0.8$ Gbps and $\sigma^2(T) = 0.05$) and link-capacity assignment ($1, 10, 40$ Gbps)
- `DCR-generator` for the remaining network parameters ($L = 1500$, $n_i = l_{ij} = L/w_{ij}$, $\sigma = 3L$)
- Distributed at `http://www.di.unipi.it/optimize/Data/MMCF.html#UMMCF`
- Experiments with "unloaded networks", but "loaded" case analogous

MI-SOCP models – Cplex

	Cplex P				Cplex bM			
	avg		max		avg		max	
	t	n	t	n	t	n	t	n
abilene	0.011	0.000	0.03	0	0.02	0.03	0.09	1
atlanta	0.015	0.044	0.18	1	0.03	0.07	0.17	1
cost266	0.015	0.017	0.06	1	0.05	0.03	0.26	1
dfn-bwin	0.012	0.000	0.03	0	0.05	0.02	0.11	1
dfn-gwin	0.020	0.151	0.10	1	0.05	0.00	0.16	0
di-yuan	0.051	1.190	0.34	18	0.11	1.36	0.62	31
france	0.014	0.000	0.05	0	0.04	0.02	0.16	1
geant	0.011	0.016	0.06	1	0.03	0.03	0.19	1
germany50	0.024	0.025	0.10	1	0.09	0.06	0.70	1
giul39	0.245	0.547	0.99	13	1.27	15.33	6.68	610
india35	0.021	0.036	0.27	1	0.08	0.07	0.58	4
janos-us	0.093	0.108	0.63	7	0.43	2.65	1.55	30
janos-us-ca	0.141	0.138	0.83	8	0.80	5.76	2.76	243
newyork	0.018	0.034	0.14	1	0.07	0.05	0.28	1
nobel-eu	0.016	0.009	0.08	1	0.04	0.05	0.26	1
nobel-ger	0.011	0.020	0.04	1	0.04	0.08	0.24	3

MI-SOCP models – Cplex (cont.)

nobel-us	0.015	0.083	0.10	1	0.04	0.04	0.19	1
norway	0.035	0.079	0.32	8	0.11	0.36	0.96	8
pdh	0.042	0.444	0.38	8	0.11	0.74	0.38	13
pioro40	0.019	0.039	0.27	1	0.10	0.14	0.57	6
polska	0.020	0.042	0.11	1	0.03	0.08	0.09	1
sun	0.165	0.587	0.89	13	0.65	7.68	2.36	257
ta2	0.020	0.015	0.13	1	0.12	0.08	0.89	4
garr 1999-01	0.022	0.017	0.13	1	0.09	0.21	0.33	1
garr 1999-04	0.029	0.000	0.07	0	0.10	0.07	0.45	3
garr 1999-05	0.029	0.004	0.09	1	0.10	0.08	0.40	3
garr 2001-09	0.030	0.000	0.10	0	0.11	0.10	0.44	3
garr 2001-12	0.029	0.000	0.08	0	0.09	0.16	0.32	3
garr 2004-04	0.028	0.000	0.18	0	0.09	0.05	0.31	3
garr 2009-08	0.087	0.005	0.46	2	0.57	0.47	1.99	27
garr 2009-09	0.089	0.011	0.62	4	0.60	0.61	2.19	36
garr 2009-12	0.090	0.013	0.78	4	0.60	0.59	2.47	44
garr 2010-01	0.093	0.013	0.50	4	0.61	0.57	2.32	32
w1-100-04	1.854	3.176	43.14	85	8.88	164.49	43.87	2585
w1-200-04	24.231	25.366	413.95	4075	231.09	2714.68	9088.54	127429

MI-SOCP models – GUROBI

	GUROBI P				GUROBI bM			
	avg		max		avg		max	
	t	n	t	n	t	n	t	n
abilene	0.011	0.0	0.03	0	0.032	0.1	0.06	3
atlanta	0.012	0.5	0.03	8	0.044	1.6	0.08	15
cost266	0.012	0.4	0.05	11	0.099	0.8	0.30	27
dfn-bwin	0.007	0.0	0.01	0	0.068	0.0	0.08	0
dfn-gwin	0.017	0.0	0.04	0	0.104	0.1	0.31	4
di-yuan	0.028	2.0	0.21	46	0.116	4.9	0.46	74
france	0.011	0.3	0.03	6	0.079	1.2	0.18	17
geant	0.011	0.7	0.04	11	0.062	1.2	0.17	22
germany50	0.016	1.1	0.26	34	0.166	2.5	0.93	52
giul39	0.424	67.6	6.69	1308	1.795	138.5	30.02	2212
india35	0.014	0.4	0.12	14	0.132	1.8	0.34	29
janos-us	0.150	21.2	2.14	767	0.717	85.4	16.54	1168
janos-us-ca	0.285	47.1	7.87	916	1.741	158.4	25.93	1595
newyork	0.013	0.8	0.04	14	0.091	2.2	0.22	22
nobel-eu	0.013	0.2	0.09	9	0.080	0.4	0.25	31
nobel-ger	0.012	0.4	0.04	11	0.056	1.4	0.33	38

MI-SOCP models – GUROBI (cont.)

nobel-us	0.012	0.8	0.05	11	0.047	0.9	0.15	11
norway	0.033	2.8	0.44	30	0.141	7.7	0.63	55
pdh	0.023	4.6	0.09	47	0.081	7.1	0.23	45
pioro40	0.015	0.6	0.09	13	0.160	2.6	0.57	44
polska	0.010	0.5	0.03	7	0.038	1.2	0.06	9
sun	0.189	39.6	0.76	282	0.961	126.9	5.68	583
ta2	0.018	0.6	0.12	27	0.214	1.9	1.52	33
garr 1999-01	0.034	0.5	0.09	9	0.096	6.6	0.38	17
garr 1999-04	0.016	1.9	0.11	26	0.115	2.7	0.55	35
garr 1999-05	0.018	2.0	0.08	25	0.139	3.5	0.79	36
garr 2001-09	0.020	2.0	0.09	19	0.156	4.0	0.97	29
garr 2001-12	0.015	0.0	0.04	0	0.116	0.1	0.31	17
garr 2004-04	0.021	3.0	0.06	14	0.128	3.5	0.57	27
garr 2009-08	0.070	7.6	0.72	124	0.776	18.8	5.39	164
garr 2009-09	0.071	7.6	0.59	202	0.918	21.8	4.85	212
garr 2009-12	0.071	7.6	0.55	123	0.920	22.7	6.21	352
garr 2010-01	0.073	7.6	0.68	114	0.916	22.8	5.76	339
w1-100-04	2.372	159.3	7.09	703	14.064	407.2	110.36	5339
w1-200-04	9.575	241.4	63.37	1395	134.145	637.0	2384.84	10943

Outline

MI-SOCP Model for WRP

- $\theta_{ij} = L/r_{ij} + |P(i,j)|L/w_{ij} \approx (11) \Longrightarrow$ (basically) same model
- But requires access control: not to make existing flows unfeasible

MI-SOCP Model for WRP

- $\theta_{ij} = L/r_{ij} + |P(i,j)|L/w_{ij} \approx (11) \implies$ (basically) same model
- But requires access control: not to make existing flows unfeasible
- Delay slack:
$$\bar{\delta}^k = \delta^k - \frac{\sigma^k}{r^k_{min}} - \sum_{(i,j)\in P^k} \left(\frac{L}{r^k_{ij}} + |P(i,j)|\frac{L}{w_{ij}} + l_{ij} + n_i \right)$$
- Access control constraint, one for each $k \in K$
$$\sum_{(i,j)\in P^k} \frac{L}{w_{ij}} x_{ij} \leq \bar{\delta}^k$$
$|P(i,j)|$ increases by one in all (i,j) that the new path traverses
- Can be used to "preprocess away" some arcs
- The coefficients are the same for each flow, can use path (+ RHS) dominance to detect redundant ones
- Still, possibly many constraints ($|K| \approx n^2$)

MI-SOCP Model for FB

- $\theta_{ij} = L/r_{ij} + (\,|P(i,j)| + \phi(r_{ij})\,)L/w_{ij}$ ($\approx$ WRP), where

$$\phi(r) = (w_{ij} - r)/\min\{\,r\,,\,r_{ij}^{min}\,\}$$

MI-SOCP Model for FB

- $\theta_{ij} = L/r_{ij} + (\,|P(i,j)| + \phi(r_{ij})\,)L/w_{ij}$ ($\approx$ WRP), where
$$\phi(r) = (w_{ij} - r)/\min\{\,r\,,\,r_{ij}^{min}\,\}$$
- Since r_{ij}^{min} is fixed, can be rewritten as
$$\phi(r) = \begin{cases} \phi_1(r) = w_{ij}/r - 1 & \text{if } 0 < r \leq r_{ij}^{min} \\ \phi_2(r) = (w_{ij} - r)/r_{ij}^{min} & \text{if } r_{ij}^{min} \leq r \leq c_{ij}\,(\leq w_{ij}) \end{cases}$$
- Convex!: both ϕ_1 and ϕ_2 are, and $\phi_1'(r_{ij}^{min}) \leq \phi_2'(r_{ij}^{min})$

MI-SOCP Model for FB

- $\theta_{ij} = L/r_{ij} + (\,|P(i,j)| + \phi(r_{ij})\,)L/w_{ij}$ ($\approx$ WRP), where
$$\phi(r) = (w_{ij} - r)/\min\{\, r \,,\, r_{ij}^{min} \,\}$$
- Since r_{ij}^{min} is fixed, can be rewritten as
$$\phi(r) = \begin{cases} \phi_1(r) = w_{ij}/r - 1 & \text{if } 0 < r \leq r_{ij}^{min} \\ \phi_2(r) = (w_{ij} - r)/r_{ij}^{min} & \text{if } r_{ij}^{min} \leq r \leq c_{ij}\,(\leq w_{ij}) \end{cases}$$
- Convex!: both ϕ_1 and ϕ_2 are, and $\phi_1'(r_{ij}^{min}) \leq \phi_2'(r_{ij}^{min})$
- Can use the classical variable splitting reformulation:
$$\phi(r) = \phi_1(r') + \phi_2(r'' + r_{ij}^{min}) - \phi(r_{ij}^{min}) \qquad \text{s.t.}$$
$0 \leq r_{ij}' \leq r_{ij}^{min}$, $0 \leq r_{ij}'' \leq (c_{ij} - r_{ij}^{min})$, $r = r' + r''$
- The idea: r' is cheaper than r'' (ϕ' is nondecreasing) and hence it gets used first $\Longrightarrow r'' > 0$ only if $r' = r_{ij}^{min}$

MI-SOCP Model I for FB

- Note that $\phi_1(r') + \phi_1(r'' + r_{ij}^{min}) - \phi(r_{ij}^{min}) = w_{ij}/r' - r''/r_{ij}^{min} - 1 \Longrightarrow$

$$\theta_{ij} = v_{ij} + v'_{ij} + \frac{L}{w_{ij}} \left[\, (\, |P(i,j)| - 1 \,) x_{ij} - \frac{r''_{ij}}{r_{ij}^{min}} \, \right]$$

$$r_{ij} = r'_{ij} + r''_{ij} \quad , \quad \rho x_{ij} \leq r'_{ij} \leq r_{ij}^{min} x_{ij} \quad , \quad 0 \leq r''_{ij} \leq (c_{ij} - r_{ij}^{min}) x_{ij}$$

$$v_{ij} r_{ij} \geq L x_{ij}^2 \;\;,\;\; v_{ij} \geq 0 \;\;,\;\; v'_{ij} r'_{ij} \geq L x_{ij}^2 \;\;,\;\; v'_{ij} \geq 0$$

still compact, but two conic constraints to represent the same L/r_{ij}, one for $r_{ij} \leq r_{ij}^{min}$, and the other for r_{ij} "unconstrained"

MI-SOCP Model I for FB

- Note that $\phi_1(r') + \phi_1(r'' + r_{ij}^{min}) - \phi(r_{ij}^{min}) = w_{ij}/r' - r''/r_{ij}^{min} - 1 \Longrightarrow$

$$\theta_{ij} = v_{ij} + v'_{ij} + \frac{L}{w_{ij}} \left[(\,|P(i,j)| - 1\,)x_{ij} - \frac{r''_{ij}}{r_{ij}^{min}} \right]$$

$$r_{ij} = r'_{ij} + r''_{ij} \quad , \quad \rho x_{ij} \leq r'_{ij} \leq r_{ij}^{min} x_{ij} \quad , \quad 0 \leq r''_{ij} \leq (c_{ij} - r_{ij}^{min})x_{ij}$$

$$v_{ij} r_{ij} \geq L x_{ij}^2 \ , \ v_{ij} \geq 0 \ , \ v'_{ij} r'_{ij} \geq L x_{ij}^2 \ , \ v'_{ij} \geq 0$$

  still compact, but two conic constraints to represent the same L/r_{ij}, one for $r_{ij} \leq r_{ij}^{min}$, and the other for r_{ij} "unconstrained"

- Can we do better? Consider that
  - $\phi_1(r_{ij}^{min}) = \phi_2(r_{ij}^{min})$
  - ϕ'_2 is constant while ϕ'_1 is strictly increasing
  - $\phi_1(w_{ij}) = \phi_2(w_{ij}) = 0$

  $\Longrightarrow \phi_2(r) \geq \phi_1(r)$ for $r \in [r_{ij}^{min}, w_{ij}]$, $\phi_1(r) \geq \phi_2(r)$ for $r \in (0, r_{ij}^{min}]$

  $\Longrightarrow \phi(r) = \max\{\, \phi_1(r)\ ,\ \phi_2(r)\,\}$!

MI-SOCP Model II for FB

- Can use the "cutting planes" representation of ϕ

$$v \geq \phi_1(r) = w_{ij}/r - 1 \qquad , \qquad v \geq \phi_2(r) = (w_{ij} - r)/r_{ij}^{min}$$

- Alternative formulation (recall the L/w_{ij} factor):

$$\theta_{ij} = v_{ij} + v'_{ij} + \frac{L}{w_{ij}}(|P(i,j)| + 1)x_{ij}$$

$$v_{ij} r_{ij} \geq L x_{ij}^2 \quad , \quad v_{ij} \geq 0$$

$$v'_{ij} \geq v_{ij} - L/w_{ij}$$

$$v'_{ij} \geq (L/r_{ij}^{min}) x_{ij} - L r_{ij}/(w_{ij} r_{ij}^{min})$$

only one conic constraint, less variables

- Note the $x_{ij} \cdot w_{ij}/r_{ij}^{min}$ in ϕ_2: otherwise, $v'_{ij} \geq L/r_{ij}^{min}$ even if $x_{ij} = 0$

Admission Control for FB

- "Abstract" admission control constraint for FB: same $\bar{\delta}^k$ as WRP,

$$\sum_{(i,j)\in P^k} \frac{L}{w_{ij}} \left(x_{ij} + \frac{w_{ij} - r_{ij}^k}{\min\{\, r_{ij} \,,\, r_{ij}^{min} \,\}} \right) \leq \bar{\delta}^k$$

$|P(i,j)| \mathrel{+}= 1$, plus r_{ij}^{min} decreases $\Longleftrightarrow r_{ij} \leq r_{ij}^{min}$

Admission Control for FB

- "Abstract" admission control constraint for FB: same $\bar{\delta}^k$ as WRP,

$$\sum_{(i,j)\in P^k} \frac{L}{w_{ij}} \left(x_{ij} + \frac{w_{ij} - r_{ij}^k}{\min\{\, r_{ij} \,,\, r_{ij}^{min} \,\}} \right) \leq \bar{\delta}^k$$

  $|P(i,j)| \mathrel{+}= 1$, plus r_{ij}^{min} decreases $\Longleftrightarrow r_{ij} \leq r_{ij}^{min}$

- Extra term $(w_{ij} - r_{ij}^k)/r_{ij}$, but only if $r_{ij} \leq r_{ij}^{min}$; otherwise, constant term $(w_{ij} - r_{ij}^k)/r_{ij}^{min} \quad \Longrightarrow$

$$\sum_{(i,j)\in P^k} \frac{L}{w_{ij}} \Big(x_{ij} + (w_{ij} - r_{ij}^k) z_{ij} \Big) \leq \bar{\delta}^k$$

$$s_{ij} \leq r_{ij} \;\;,\;\; s_{ij} \leq r_{ij}^{min} \;\;,\;\; s_{ij} z_{ij} \geq x_{ij}^2 \;\;,\;\; z_{ij} \geq 0$$

  $+2|A|$ variables, $+|A|$ conic constraints but shared among flows

- Different coefficients (to share the z_{ij}), dominance more difficult
- Arc-based preprocessing still possible (using $r_{ij} = c_{ij}$)

Computational results for WRP and FB

Er . . . , not ready yet, sorry!

- Still brewing, too early to post tables
- So far good enough; usually within a small factor of running time
- But there are exceptions, especially FB w.r.t. SRP:
  can see a factor of 50 in max time, a factor of 10 in average time
- Admittedly unrefined tests, but can we do better?

Outline

Combinatorial properties

- A MINLP with strong network structure, how to exploit it?

Combinatorial properties

- A MINLP with strong network structure, how to exploit it?
- $\sigma = L = 0 \Longrightarrow r_{ij} = \rho x_{ij} \Longrightarrow$ Constrained Shortest Path (CSP) (this gives more than one idea, and proves $\mathcal{NP}$-hardness)

Combinatorial properties

- A MINLP with strong network structure, how to exploit it?
- $\sigma = L = 0 \Longrightarrow r_{ij} = \rho x_{ij} \Longrightarrow$ Constrained Shortest Path (CSP) (this gives more than one idea, and proves $\mathcal{NP}$-hardness)
- Feasibility is easy: delay $\searrow$ when $r_{ij} \nearrow \Longrightarrow r_{ij} = c_{ij} \Longrightarrow$ modified arc costs $\bar{l}_{ij} = L/c_{ij} + (\, l'_{ij} = L/w_{ij} + l_{ij} + n_i \,)$

Combinatorial properties

- A MINLP with strong network structure, how to exploit it?
- $\sigma = L = 0 \Longrightarrow r_{ij} = \rho x_{ij} \Longrightarrow$ Constrained Shortest Path (CSP) (this gives more than one idea, and proves $\mathcal{NP}$-hardness)
- Feasibility is easy: delay $\searrow$ when $r_{ij} \nearrow \Longrightarrow r_{ij} = c_{ij} \Longrightarrow$ modified arc costs $\bar{l}_{ij} = L / c_{ij} + (\, l'_{ij} = L / w_{ij} + l_{ij} + n_i \,)$
- But using (i,j) with "low" $c_{ij} \searrow r_{min} \Longrightarrow \nearrow$ the delay: $G^r = (N, A^r)$ with $A^r = \{\, (i,j) \in A \;:\; c_{ij} \geq r \,\} \Longrightarrow r_{min} \geq r$
- For each $r \in C = \{\, c_{ij} \;:\; (i,j) \in A \,\}$:
  - solve s–d shortest path P on G^r w.r.t. $\bar{l}$
  - if $\bar{l}(P) \leq \delta - \sigma / r$, then P feasible: stop
  - if no feasible P found, then problem unfeasible (for fixed P, both LHS and RHS of (4) increase with r)
- Keep f-best solution found: ERA-I heuristic

Equal-Cost, Equal Rate Allocation

- Equal Rate Allocation: $r_{ij} = r$ $(\geq \rho)$ for all $(i,j) \in P$ $(\Longrightarrow r_{min} = r)$

Equal-Cost, Equal Rate Allocation

- Equal Rate Allocation: $r_{ij} = r$ ($\geq \rho$) for all $(i,j) \in P$ ($\Longrightarrow r_{min} = r$)
- EC-ERA-SRP-SFSP-DCR ($f_{ij} = 1$) is easy for fixed r:
  - run Bellman-Ford on G^r with costs $l^r_{ij} = L/r + l'_{ij}$
  - at each round of BF, check path P entering d (if any)
  - if $l^r(P) \leq \delta - \sigma/r$ then stop: P optimal

Equal-Cost, Equal Rate Allocation

- Equal Rate Allocation: $r_{ij} = r$ $(\geq \rho)$ for all $(i, j) \in P$ $(\Longrightarrow r_{min} = r)$
- EC-ERA-SRP-SFSP-DCR $(f_{ij} = 1)$ is easy for fixed r:
  - run Bellman-Ford on G^r with costs $l^r_{ij} = L/r + l'_{ij}$
  - at each round of BF, check path P entering d (if any)
  - if $l^r(P) \leq \delta - \sigma/r$ then stop: P optimal
- Works because BF solves hop-constrained shortest path: find least-cost($=$ delay) path with that number of hops, but r fixed $\Longrightarrow$ true cost proportional to $|P|$
- Each round, cost($=$ delay) $\searrow$ but hop count ($=$ cost) $\nearrow$: first feasible path is optimal
- Repeating the above for all $r \in C$ does not solve $(\ldots)$DCR; counterexample: returned path P with delay constraint not tight

$$\Delta(r, P) = \frac{\sigma + L|P|}{r} + \sum_{(i,j)\in P} l'_{ij} < \delta$$

Equal-Cost, Equal Rate Allocation

- Obvious solution: for each feasible P reduce r until constraint tight

$$\tilde{r}(P) = (\,\sigma + L|P|\,)/(\,\delta - l'(P)\,)$$

$$\Longrightarrow \tilde{r}(P) \leq r,\ \Delta(\tilde{r}(P), P) = \delta \text{ (keep feasibility, improve objective)}$$

Theorem

For all $r \in C$ run BF on G^r, for all P decrease r, keep best P (don't stop at first feasible): solves EC-ERA-SRP-SFSP-DCR in $O(|C|nm) \leq O(nm^2)$

Proof.

Optimal (r^*, P^*), $\bar{r} = \min\{r \in C : r \geq r^*\}$. When $\bar{r}$ chosen, $P^* \in G^{\bar{r}}$ and delay-feasible ($r^* \leq \bar{r}$, delay $\searrow$ when $r \nearrow$) $\Longrightarrow$ BF finds minimum-delay feasible P with $h^* = |P^*|$ hops $\Longrightarrow \Delta(\bar{r}, P) \leq \Delta(\bar{r}, P^*)$; $=$ must hold $\Longrightarrow$ P optimal. In fact $\Delta(\bar{r}, P) < \Delta(\bar{r}, P^*) \Longrightarrow l'(P) < l'(P^*) \Longrightarrow$ $\delta - l'(P) > \delta - l'(P^*) \Longrightarrow \tilde{r}(P) < \tilde{r}(P^*) \Longrightarrow P$ better than P^*. $\square$

- Obviously, then, ERA-H heuristic for EC-SRP-SFSP-DCR

Improving ERA-H (and ERA-I)

- Standard Bellman-Ford is $\Omega(mn)$, slow in practice
- Alternative implementation: SPT.L.Queue with label pairs $(l, d) \equiv$ shortest path on acyclic graph $n \times G^r$, not much better

Improving ERA-H (and ERA-I)

- Standard Bellman-Ford is $\Omega(mn)$, slow in practice
- Alternative implementation: SPT.L.Queue with label pairs $(l, d) \equiv$ shortest path on acyclic graph $n \times G^r$, not much better
- Heuristic alternative: SPT.L.Queue on original G^r, each time d extracted from Q check delay of that P
- Can miss the optimal path but rarely, much faster

	mean time		max time		mean gap		max gap	
	H	HBF	H	HBF	H	HBF	H	HBF
garr 2009_08	2.8e-5	3.2e-3	1.0e-2	1.0e-2	0.001	0.000	0.386	0.000
garr 2009_12	2.4e-5	3.1e-3	1.0e-2	1.0e-2	0.001	0.000	0.240	0.000
garr 2010_01	7.0e-6	3.2e-3	1.0e-2	1.0e-2	0.001	0.000	0.241	0.000
giul39	2.0e-5	5.4e-3	1.0e-2	1.0e-2	0.011	0.000	0.570	0.427
janos-us	0.0e+0	5.8e-4	0.0e+0	1.0e-2	0.004	0.000	0.275	0.000
janos-us-ca	2.0e-5	2.1e-3	1.0e-2	1.0e-2	0.010	0.000	0.289	0.041
sun	0.0e+0	7.4e-4	0.0e+0	1.0e-2	0.008	0.001	0.431	0.431
Waxman_100	1.5e-4	1.4e-1	1.0e-2	1.8e-1	0.025	0.011	0.750	0.750
Waxman_200	5.9e-3	2.3e+0	2.0e-2	2.6e+0	0.115	0.105	0.815	0.815

Extending ERA: Non-equal but Integer f_{ij}

- Non-equal but integer f_{ij}: use dynamic programming instead of BF
- $\bar{f} \geq$ path cost, e.g. $\bar{f} = (n-1)f_{max} = \max\{ f_{ij} \,:\, (i,j) \in A \}$
- DAG $\widetilde{G} = \bar{F} \times G$: nodes (i,f) for $f \in \bar{F} = \{0, 1, \ldots, \bar{f}\}$, $(i,j) \in A$ $\Longrightarrow ((i,f),(j,f+f_{ij})) \in \widetilde{A}$ (unless $f + f_{ij} > \bar{f}$), same delay, capacity
- For $f \in \bar{F}$ and r, find the minimum-delay s–d path in G with cost equal to fr $\equiv$ visit $\widetilde{G}$, $O(\bar{f}m)$
- Adapt ERA-H as follows:
  - for each $r \in C$, $\widetilde{G}^r$ with $\widetilde{A}^r$ (= arcs with capacity $\geq r$, $|\widetilde{A}^r| \leq O(\bar{f}m)$)
  - perform BFS of $\widetilde{G}^r$ from $(s,0)$; (d,f) visited for some $f \Longrightarrow$ minimum-delay s–d path of cost f with the given number of hops
  - if P delay-feasible set $r = \tilde{r}(P)$, keep (fr)-best solution found

Theorem

If $f_{ij} \in \mathbb{N}$, then the algorithm solves ERA-SRP-SFSP-DCR in $O(|C|\bar{f}m) \leq O(nm^2 f_{max})$ (pseudo-polynomial)

Extending ERA: Non-equal Continuous f_{ij}

- NE continuous f_{ij}: standard cost rounding approximation algorithm
- Standard idea: scaling factor $f \in F = \{ f_{ij} \;:\; (i,j) \in A \}$ ($|F| \leq m$), scaled costs $\tilde{f}_{ij} = \lceil f_{ij} / K \rceil$ where $K = (\varepsilon f)/(n-1)$
- On reduced graph G_f without arcs with cost $> f$, $\tilde{f}_{ij} \leq \lceil n/\varepsilon \rceil \Longrightarrow$ the pseudo-poly algorithm is $O(n^2 m^2 / \varepsilon)$
- Algorithm: cycle over all scaling factors f, apply pseudo-poly algorithm to G_f, keep best f-solution of all these found

Theorem

The algorithm finds a ε-optimal solution for ERA-SFSP-SRP-DCR (with unscaled f_{ij}) in $O(|F| n^2 m^2 / \varepsilon)$ (Fully Poly-time Approximation Scheme)

- Proof actually quite standard, follows the same route
- Tricky part: select f "large enough" so $\tilde{f}_{ij}$ "small", but also "small enough" ($f \leq f(P^*)$); heuristically, $f = f_{max}$ ($G_f = G$) should work
- Yet, we don't really want to solve the ERA version

ERA-Based Heuristics: Experiments

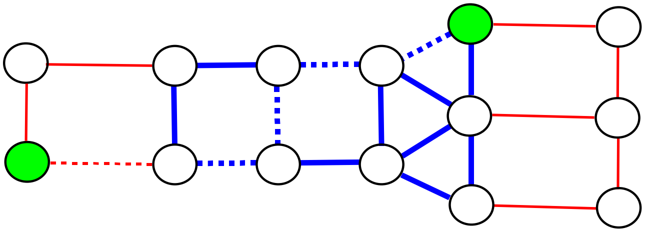
				ERA-I		ERA-H		
instance	n	m	k	avg	max	avg	max	inf
abilene	12	15	31	0.52	0.92	0.000	0.000	0.06
atlanta	15	22	45	0.57	0.88	0.000	0.000	0.07
cost266	37	57	120	0.48	0.95	0.000	0.000	0.17
dfn-bwin	10	45	45	0.03	0.06	0.000	0.000	0.00
dfn-gwin	11	47	53	0.16	0.86	0.000	0.000	0.02
di-yuan	11	42	58	0.48	0.90	0.000	0.000	0.12
france	25	45	66	0.44	0.90	0.000	0.000	0.02
geant	22	36	63	0.46	0.89	0.000	0.001	0.06
germany50	50	88	276	0.50	0.90	0.000	0.001	0.21
giul39	39	172	1482	0.67	0.97	0.011	0.570	0.10
india35	35	80	195	0.53	0.93	0.000	0.000	0.11
janos-us	26	84	650	0.71	0.95	0.004	0.275	0.18
janos-us-ca	39	122	1482	0.68	0.95	0.010	0.289	0.23
newyork	16	49	89	0.50	0.90	0.000	0.000	0.03
nobel-eu	28	41	106	0.55	0.93	0.000	0.000	0.23
nobel-ger	17	26	51	0.49	0.93	0.000	0.000	0.10

- gap with optimum, inf = feasible wrongly declared unfeasible

ERA-Based Heuristics: Experiments (cont.)

nobel-us	14	21	24	0.35	0.90	0.000	0.001	0.00
norway	27	51	341	0.71	0.94	0.000	0.000	0.12
pdh	11	34	54	0.64	0.90	0.000	0.001	0.04
pioro40	40	89	204	0.40	0.89	0.000	0.000	0.25
polska	12	18	24	0.59	0.90	0.000	0.000	0.00
sun	27	102	702	0.76	0.95	0.008	0.431	0.06
ta2	65	108	388	0.45	0.92	0.000	0.000	0.31
garr 1999-01	16	36	240	0.65	0.88	0.000	0.001	0.02
garr 1999-04	23	50	506	0.57	0.94	0.000	0.001	0.75
garr 1999-05	23	50	506	0.55	0.94	0.000	0.000	0.75
garr 2001-09	22	48	462	0.60	0.94	0.000	0.000	0.74
garr 2001-12	24	52	552	0.59	0.94	0.000	0.000	0.75
garr 2004-04	22	48	462	0.56	0.94	0.000	0.000	0.75
garr 2009-08	54	136	2862	0.65	0.94	0.001	0.386	0.85
garr 2009-09	55	138	2970	0.67	0.94	0.000	0.000	0.85
garr 2009-12	54	136	2862	0.67	0.94	0.001	0.240	0.85
garr 2010-01	54	136	2862	0.67	0.94	0.001	0.241	0.85
w1-100-04	100	414	664	0.77	0.95	0.015	0.739	0.07
w1-200-04	200	1550	1528	0.71	0.96	0.015	0.814	0.05

Why does ERA fail so often?



- Hub-and-spoke-like network with well-connected core (40/100 Gb) but weaker links to the periphery (1 Gb)
- Path from a core node to a peripheral one has to cross a weak link
- ERA has to allocate the same rate to all links $\Longrightarrow$ no more than the weak link's (residual) capacity $\Longrightarrow$ cannot meet the deadline
- The deadline can be met by reserving more capacity on core links

3-Pronged Approach

- MI-SOCP approach accurate but slow
  ERA-* approaches fast but inaccurate
- Best of both worlds: 3-pronged approach
  1. run ERA-I, if instance unfeasible terminate
  2. otherwise run ERA-H: if a solution found, report it and terminate
  3. if all else fails, then run model P and report its solution
- So crude, does it really work?

3-Pronged Approach: Experiments

Cplex				GUROBI				Gaps		ERA-H		
SOCP		3P		SOCP		3P						
avg	max	avg	max	avg	max	avg	max	avg	max	avg	max	inf
0.009	0.02	0.001	0.01	0.009	0.02	0.001	0.01	0.00	0.00		0.00	0.06
0.016	0.16	0.001	0.02	0.010	0.03	0.001	0.02	0.00	0.00		0.00	0.07
0.013	0.05	0.002	0.03	0.012	0.04	0.003	0.04	0.00	0.00		0.00	0.17
0.011	0.02	0.000	0.00	0.007	0.01	0.000	0.01	0.00	0.00		0.00	0.00
0.019	0.09	0.000	0.01	0.015	0.04	0.000	0.01	0.00	0.00		0.00	0.02
0.050	0.35	0.017	0.35	0.028	0.22	0.012	0.23	0.00	0.00		0.00	0.12
0.015	0.04	0.000	0.01	0.010	0.03	0.000	0.01	0.00	0.00		0.00	0.02
0.013	0.05	0.001	0.01	0.010	0.04	0.001	0.03	0.00	0.00		0.00	0.06
0.021	0.09	0.005	0.08	0.017	0.24	0.007	0.27	0.00	0.00	7e-5	0.01	0.21
0.254	1.01	0.019	0.66	0.449	7.57	0.087	6.52	0.01	0.57	3e-4	0.01	0.10
0.019	0.25	0.002	0.04	0.016	0.11	0.002	0.07	0.00	0.00		0.00	0.11
0.091	0.62	0.013	0.33	0.153	2.25	0.051	2.19	0.00	0.28	1e-4	0.01	0.18
0.144	0.84	0.026	0.49	0.298	9.59	0.118	7.70	0.01	0.29	2e-4	0.01	0.23
0.017	0.13	0.000	0.02	0.015	0.04	0.001	0.02	0.00	0.00		0.00	0.03
0.014	0.05	0.004	0.05	0.016	0.09	0.005	0.09	0.00	0.00		0.00	0.23
0.010	0.03	0.002	0.03	0.015	0.04	0.002	0.04	0.00	0.00		0.00	0.10

3-Pronged Approach: Experiments (cont.)

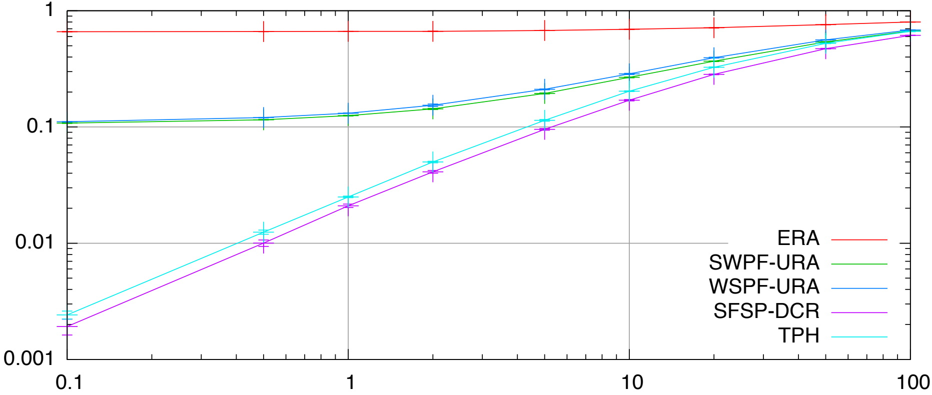
0.013	0.09	0.000	0.00	0.014	0.05	0.000	0.00	0.00	0.00		0.00	0.00
0.032	0.30	0.005	0.25	0.035	0.32	0.005	0.13	0.00	0.00	6e-5	0.01	0.12
0.034	0.30	0.001	0.02	0.026	0.10	0.002	0.10	0.00	0.00		0.00	0.04
0.019	0.27	0.007	0.25	0.018	0.09	0.007	0.09	0.00	0.00	5e-5	0.01	0.25
0.016	0.09	0.000	0.00	0.014	0.03	0.000	0.00	0.00	0.00		0.00	0.00
0.154	0.89	0.006	0.36	0.188	0.87	0.009	0.40	0.01	0.43	2e-4	0.01	0.06
0.019	0.12	0.008	0.05	0.020	0.13	0.009	0.13	0.00	0.00	8e-5	0.01	0.31
0.025	0.12	0.001	0.03	0.035	0.10	0.001	0.03	0.00	0.00	4e-5	0.01	0.02
0.030	0.08	0.022	0.06	0.017	0.12	0.016	0.10	0.00	0.00	4e-5	0.01	0.75
0.028	0.08	0.021	0.06	0.018	0.08	0.016	0.08	0.00	0.00	6e-5	0.01	0.75
0.026	0.09	0.021	0.08	0.022	0.09	0.018	0.09	0.00	0.00	4e-5	0.01	0.74
0.027	0.07	0.022	0.07	0.016	0.04	0.012	0.04	0.00	0.00	4e-5	0.01	0.75
0.026	0.17	0.020	0.05	0.022	0.06	0.019	0.06	0.00	0.00	4e-5	0.01	0.75
0.084	0.44	0.075	0.44	0.069	0.70	0.065	0.71	0.00	0.39	2e-4	0.01	0.85
0.086	0.62	0.078	0.62	0.069	0.56	0.063	0.57	0.00	0.00	2e-4	0.01	0.85
0.088	0.75	0.078	0.73	0.071	0.52	0.061	0.50	0.00	0.24	2e-4	0.01	0.85
0.087	0.46	0.076	0.45	0.074	0.61	0.066	0.59	0.00	0.24	2e-4	0.01	0.85
1.906	46.7	0.034	1.84	2.354	8.35	0.150	3.54	0.01	0.74	2e-3	0.01	0.07
23.660	357.7	0.247	54.29	9.033	63.19	0.399	12.36	0.01	0.81	1e-2	0.02	0.05

Outline

Does it really matter in practice?

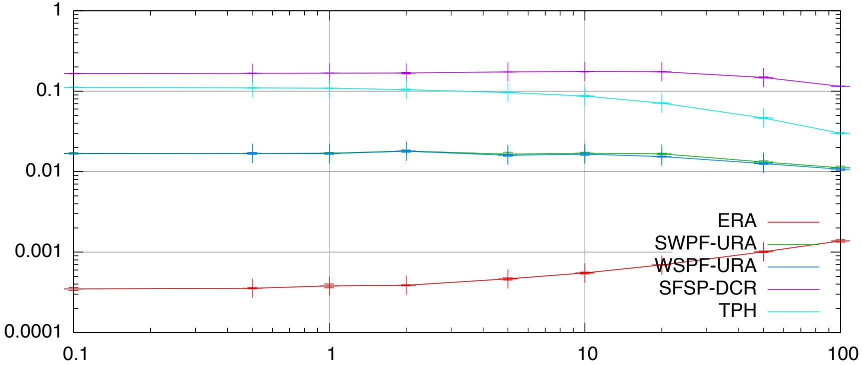
- Simulating the network behavior, large number of path computations
- Exponential interarrival (avg $= \lambda$), exponential duration (avg $=$ 1s)
- $\sigma = 3$ MTU and δ random in $[\, d_{min}\,,\, d_{min} + \beta(d_{max} - d_{min})\,]$
  $d_{min} =$ minimum feasible deadline, $d_{max} =$ delay constraint inactive
- Average of five independent replicas, and 95% confidence intervals
- Comparing all practical approaches known so far (2 new):
  1. ERA (equal rate allocation)
  2. SWPF-URA: shortest-widest-path + optimal (unequal) rate allocation
  3. WSPF-URA: widest-shortest-path + optimal (unequal) rate allocation
  4. SFSP-DCR: MI-SOCP model (perspective version)
  5. TPH: 3-pronged heuristic
- Same real-world topologies, realistic capacities

Simulation results: blocking probability



- ERA fails far too much (allocating the same rate a bad idea)
- both ERA and $*$-URA perform considerably worse than SFSP-DCR
- TPH performs quite close to the optimum
- Similar on all topologies, $\sigma \in \{1, 3, 10\}$MTU, $\beta \in \{0.2, 0.5, 1.0\}$

Simulation results: time



- SFSP-DCR slower but still affordable
- TPH much faster and almost as good
- “large” networks: $|N| = 70+$, $|A| = 230+$
- Path Computation Element makes this technically feasible

Outline

YAMV: Yet Another Modeling Variant

- Small remaining nuisance: "big" c_{max} in (8)
- Serves to define r_{min}, which serves for σ / r_{min} term

YAMV: Yet Another Modeling Variant

- Small remaining nuisance: "big" c_{max} in (8)
- Serves to define r_{min}, which serves for σ / r_{min} term
- Simple idea: $v \geq \sigma / r_{ij}$ for all $(i,j) \Longrightarrow v \geq \sigma / r_{min}$
- Alternative version of (9):

$$(\sigma + L)s_{hk} + \sum_{(i,j) \in A \setminus \{(h,k)\}} L s_{ij} + \sum_{(i,j) \in A} l'_{ij} x_{ij} \leq \delta \qquad (h,k) \in A \qquad (19)$$

$$s_{ij} r_{ij} \geq x_{ij}^2 \ , \ \ s_{ij} \geq 0 \qquad (i,j) \in A$$

$|A| - 1$ more linear constraints, one less variable and conic constraint

- Most constraints will not be active at optimum (e.g., $r_{hk} = 0 \Longrightarrow s_{hk} = 0$), plus they are linear $\Longrightarrow$ lazy constraints

Can YAMV Be a Good Idea?

- In principle yes
- Funny thing: in the optimum of the continuous relaxation, r_{min} can be $> r_{ij}$!

Can YAMV Be a Good Idea?

- In principle yes
- Funny thing: in the optimum of the continuous relaxation, r_{min} can be $> r_{ij}$!
- Example: 2 nodes, two identical parallel arcs, $\rho = f = l + n = 1$, $L = c_{max} = 10$, $\sigma = 3$, $\delta = 5$
- Optimal solution of standard continuous relaxation: $x^* = 1/2$, $r^* = (\sqrt{34} - 3)/2 \approx 1.41548$ on both arcs, $r^*_{min} = (\sqrt{34} + 7)/2 \approx 6.41548$, optimum $2r^* \approx 2.83095$
- Trick is that c_{max} large, and $r_{min} \leq r_{ij} + c_{max}(1 - x_{ij}) > r_{ij}$
- Optimal solution of YAMV continuous relaxation: $x^* = 1/2$, $r^* = 23/16 = 1.4375$, optimum $23/8 = 2.875$ (true optima 3.25)
- So it may work. Does it?

Computational results for YAMV

Computational results for YAMV

Er . . . , not ready yet, sorry!

- Still brewing, too early to post tables
- So far not promising: most often slower than traditional approach
- Still not entirely clear why, but does not seem to improve bound much & lazy constraint help but not enough

Lagrangian approaches, Take I

- A different use for YAMV: Lagrangian relaxation of (19) w.r.t. λ_{hk}

$$\min \left\{ \textstyle\sum_{(i,j)\in A} f_{ij} r_{ij} + l_{ij}(\lambda) x_{ij} + \frac{L_{ij}(\lambda) x_{ij}^2}{r_{ij}} \ : \ (6) \ , \ (7) \right\} - \delta \sum_{(h,k)\in A} \lambda_{hk}$$

where $l_{ij}(\lambda) = \bar{l}_{ij} \sum_{(h,k)\in A} \lambda_{hk}$, $L_{ij}(\lambda) = L \sum_{(h,k)\in A} \lambda_{hk} + \sigma \lambda_{ij}$.

Lagrangian approaches, Take I

- A different use for YAMV: Lagrangian relaxation of (19) w.r.t. λ_{hk}

$$\min \left\{ \sum_{(i,j)\in A} f_{ij} r_{ij} + l_{ij}(\lambda) x_{ij} + \frac{L_{ij}(\lambda) x_{ij}^2}{r_{ij}} \;:\; (6)\;,\;(7) \right\} - \delta \sum_{(h,k)\in A} \lambda_{hk}$$

where $l_{ij}(\lambda) = \bar{l}_{ij} \sum_{(h,k)\in A} \lambda_{hk}$, $L_{ij}(\lambda) = L \sum_{(h,k)\in A} \lambda_{hk} + \sigma \lambda_{ij}$.

- Reduces to a shortest path because r_{ij} can be "projected away": $x_{ij} = 0 \Longrightarrow r_{ij} = 0$, $x_{ij} = 1 \Longrightarrow r_{ij}$ solves

$$[\, l_{ij}(\lambda) +] \;\; \min \left\{ \phi(r_{ij}) = f_{ij} r_{ij} + L_{ij}(\lambda)/r_{ij} \;:\; \rho \le r_{ij} \le c_{ij} \right\}$$

that has the closed-formula expression

$$r_{ij}^*(\lambda) = \begin{cases} c_{ij} & \text{if } f_{ij} \le 0 \\ \min\left\{ c_{ij}\,, \max\left\{ \rho\,,\, \sqrt{L_{ij}(\lambda)/f_{ij}} \right\} \right\} & \text{otherwise} \end{cases}$$

Lagrangian approaches, Take I

- A different use for YAMV: Lagrangian relaxation of (19) w.r.t. λ_{hk}

$$\min \left\{ \textstyle\sum_{(i,j)\in A} f_{ij} r_{ij} + l_{ij}(\lambda) x_{ij} + \frac{L_{ij}(\lambda) x_{ij}^2}{r_{ij}} \;:\; (6)\;,\;(7) \right\} - \delta \sum_{(h,k)\in A} \lambda_{hk}$$

where $l_{ij}(\lambda) = \bar{l}_{ij} \sum_{(h,k)\in A} \lambda_{hk}$, $L_{ij}(\lambda) = L \sum_{(h,k)\in A} \lambda_{hk} + \sigma \lambda_{ij}$.

- Reduces to a shortest path because r_{ij} can be "projected away": $x_{ij} = 0 \Longrightarrow r_{ij} = 0$, $x_{ij} = 1 \Longrightarrow r_{ij}$ solves

$$[\, l_{ij}(\lambda) +]\;\; \min \left\{ \phi(r_{ij}) = f_{ij} r_{ij} + L_{ij}(\lambda)/r_{ij} \;:\; \rho \leq r_{ij} \leq c_{ij} \right\}$$

that has the closed-formula expression

$$r^*_{ij}(\lambda) = \begin{cases} c_{ij} & \text{if } f_{ij} \leq 0 \\ \min \left\{ c_{ij} \,, \max \left\{ \rho \,,\, \sqrt{L_{ij}(\lambda)/f_{ij}} \right\} \right\} & \text{otherwise} \end{cases}$$

- Use a Bundle method to find λ^*, Lagrangian heuristic, B&B, ...

Lagrangian approaches, Take II

- Alternative: original formulation, relax (10) w.r.t. unique λ

$$\min\left\{\frac{\lambda\sigma}{r_{min}}+\sum_{(i,j)\in A} f_{ij}r_{ij}+\lambda\bar{l}_{ij}x_{ij}+\frac{\lambda L x_{ij}^2}{r_{ij}} \;:\; (6)\,,\,(7)\,,\,(8)\right\}\;[-\lambda\delta\,]$$

solvable as above (one shortest path) if r_{min} fixed

Lagrangian approaches, Take II

- Alternative: original formulation, relax (10) w.r.t. unique λ

$$\min \left\{ \frac{\lambda\sigma}{r_{min}} + \sum_{(i,j)\in A} f_{ij} r_{ij} + \lambda \bar{l}_{ij} x_{ij} + \frac{\lambda L x_{ij}^2}{r_{ij}} \; : \; (6)\ ,\ (7)\ ,\ (8) \right\} \quad [-\lambda\delta\,]$$

  solvable as above (one shortest path) if r_{min} fixed

- Lagrange, then Benders': for fixed λ, find the best r_{min}

Lagrangian approaches, Take II

- Alternative: original formulation, relax (10) w.r.t. unique λ

$$\min \left\{ \frac{\lambda\sigma}{r_{min}} + \sum_{(i,j)\in A} f_{ij} r_{ij} + \lambda \bar{l}_{ij} x_{ij} + \frac{\lambda L x_{ij}^2}{r_{ij}} \; : \; (6)\;,\;(7)\;,\;(8) \right\} \quad [\,-\lambda\delta\,]$$

  solvable as above (one shortest path) if r_{min} fixed

- Lagrange, then Benders': for fixed λ, find the best r_{min}
- Of course, Benders', then Lagrange: for fixed r_{min}, find the best λ

Lagrangian approaches, Take II

- Alternative: original formulation, relax (10) w.r.t. unique λ

$$\min \left\{ \frac{\lambda\sigma}{r_{min}} + \sum_{(i,j)\in A} f_{ij} r_{ij} + \lambda \bar{l}_{ij} x_{ij} + \frac{\lambda L x_{ij}^2}{r_{ij}} \ : \ (6)\ ,\ (7)\ ,\ (8) \right\} \ [-\lambda\delta]$$

  solvable as above (one shortest path) if r_{min} fixed

- Lagrange, then Benders': for fixed λ, find the best r_{min}
- Of course, Benders', then Lagrange: for fixed r_{min}, find the best λ
- Two nested one-dimensional optimizations: hopefully very efficient
- Path are produced, can be checked for feasibility $\Longrightarrow$ heuristic (Benders', then Lagrange suspiciously similar to ERA)
- Good lower bound, hopefully faster than with SOCP solver
- B&B, etc.: lots of work, but hopefully faster

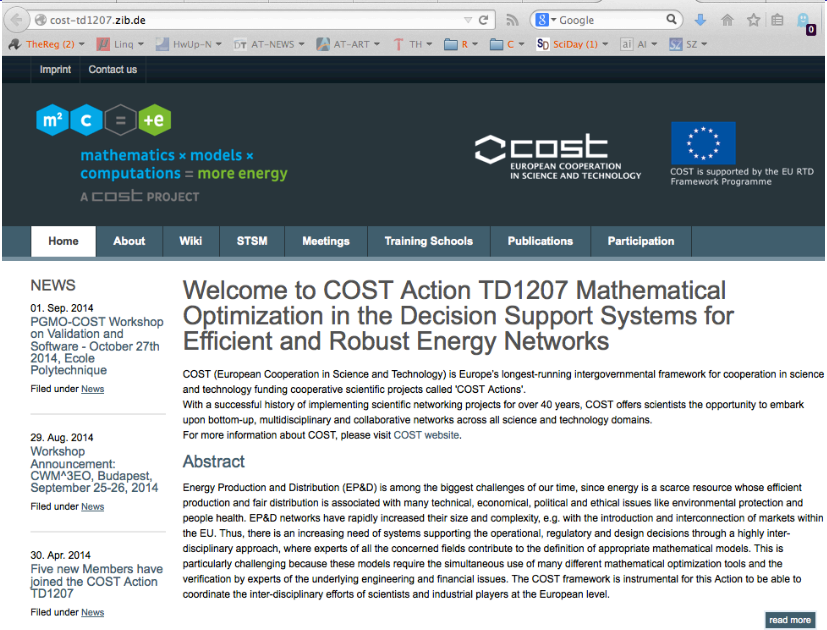
Computational results for the Lagrangian Approaches

Computational results for the Lagrangian Approaches

...in a year from now, at best ...

Outline

Commercial I: COST Action TD1207


cost-td1207.zib.de

Imprint | Contact us

mathematics × models × computations = more energy

A COST PROJECT

COST EUROPEAN COOPERATION IN SCIENCE AND TECHNOLOGY

COST is supported by the EU RTD Framework Programme

Home | About | Wiki | STSM | Meetings | Training Schools | Publications | Participation

NEWS

01. Sep. 2014
PGMO-COST Workshop on Validation and Software - October 27th 2014, Ecole Polytechnique
Filed under News

29. Aug. 2014
Workshop Announcement: CWM^3EO, Budapest, September 25-26, 2014
Filed under News

30. Apr. 2014
Five new Members have joined the COST Action TD1207
Filed under News

Welcome to COST Action TD1207 Mathematical Optimization in the Decision Support Systems for Efficient and Robust Energy Networks

COST (European Cooperation in Science and Technology) is Europe's longest-running intergovernmental framework for cooperation in science and technology funding cooperative scientific projects called 'COST Actions'.
With a successful history of implementing scientific networking projects for over 40 years, COST offers scientists the opportunity to embark upon bottom-up, multidisciplinary and collaborative networks across all science and technology domains.
For more information about COST, please visit COST website.

Abstract

Energy Production and Distribution (EP&D) is among the biggest challenges of our time, since energy is a scarce resource whose efficient production and fair distribution is associated with many technical, economical, political and ethical issues like environmental protection and people health. EP&D networks have rapidly increased their size and complexity, e.g. with the introduction and interconnection of markets within the EU. Thus, there is an increasing need of systems supporting the operational, regulatory and design decisions through a highly inter-disciplinary approach, where experts of all the concerned fields contribute to the definition of appropriate mathematical models. This is particularly challenging because these models require the simultaneous use of many different mathematical optimization tools and the verification by experts of the underlying engineering and financial issues. The COST framework is instrumental for this Action to be able to coordinate the inter-disciplinary efforts of scientists and industrial players at the European level.

read more

Commercial II: CWM^{3}O – Budapest, September 25–26

COST EUROPEAN COOPERATION IN SCIENCE AND TECHNOLOGY

HOME HOST CITY TRAVEL INFORMATION VENUE ACCOMODATION REGISTRATION PROGRAM

Workshop on Mathematical Models and Methods for Energy Optimization (CWM^{3}EO)

Within the EU COST-Action TD1207 on "Mathematical Optimization in the Decision Support Systems for Efficient and Robust Energy Networks (ICT)" the next workshop will be held at the Budapest University of Technology and Economics on **September 25-26, 2014**.

Important Dates

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical “steady state” flows
- Relevant for applications, apparently good results

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical "steady state" flows
- Relevant for applications, apparently good results
- MI-SOCP with substantial network structure =
  prototypical blend of nonlinear and combinatorial optimization

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical "steady state" flows
- Relevant for applications, apparently good results
- MI-SOCP with substantial network structure =
  prototypical blend of nonlinear and combinatorial optimization
- MINLP techniques useful (Perspective Reformulation, SOCP, . . .)
- Combinatorial techniques useful (shortest paths, dynamic programming, approximation algorithms, Lagrange, Benders', . . .)

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical "steady state" flows
- Relevant for applications, apparently good results
- MI-SOCP with substantial network structure =
  prototypical blend of nonlinear and combinatorial optimization
- MINLP techniques useful (Perspective Reformulation, SOCP, ...)
- Combinatorial techniques useful (shortest paths, dynamic programming, approximation algorithms, Lagrange, Benders', ...)
- Both are needed

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical "steady state" flows
- Relevant for applications, apparently good results
- MI-SOCP with substantial network structure = prototypical blend of nonlinear and combinatorial optimization
- MINLP techniques useful (Perspective Reformulation, SOCP, . . .)
- Combinatorial techniques useful (shortest paths, dynamic programming, approximation algorithms, Lagrange, Benders', . . .)
- Both are needed
- Still lots of work to do (multi-flow, multi-path, network design, robust, . . .), problems look pretty hard

Conclusions

- The world is indeed nonlinear, but surprisingly often nicely convex
- DCR: interesting generalization of classical "steady state" flows
- Relevant for applications, apparently good results
- MI-SOCP with substantial network structure = prototypical blend of nonlinear and combinatorial optimization
- MINLP techniques useful (Perspective Reformulation, SOCP, ...)
- Combinatorial techniques useful (shortest paths, dynamic programming, approximation algorithms, Lagrange, Benders', ...)
- Both are needed
- Still lots of work to do (multi-flow, multi-path, network design, robust, ...), problems look pretty hard
- Lots of fun. Join in! :-)

Homework – Models

1. A canonical conic constraint has the form $x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}$; write the rotated cone constraint $vs \geq Lx^2$, with $v, s \geq 0$ in canonical form
2. Write a MI-SOCP model of SFSP-SRP-DCR for the formula
$$\theta_{ij} = \frac{L}{w_{ij}} + \begin{cases} L / r_{ij}^{eff} & \text{if } P(i,j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$
Do you need access control? If so, be sure to include it
3. Write a MI-SOCP model of SFSP-WRP-DCR for the formula
$$\theta_{ij} = \Big(|P(i,j)| + 1 \Big) \frac{L}{w_{ij}} + \frac{L}{r_{ij}^{eff}}$$
You do need access control, so be sure to include it
4. As before for SFSP-FB-DCR with the worst-case formula
$$\theta_{ij} = \left(|P(i,j)| + \frac{\bar{r}_{ij} + r_{ij}}{r_{ij}^{min}} \right) \frac{L}{w_{ij}} + \frac{L}{r_{ij}^{eff}}$$

Homework – Models (cont.d)

5 [NASTY] Develop a MI-SOCP model of SFSP-DCR for the group-based approximation of SRP

$$\theta_{ij} = 3\frac{2^{\lceil \log_2 w_{ij} L / r_{ij} \rceil}}{w_{ij}} + 2\frac{L}{w_{ij}}$$

Homework – Models (cont.d)

5 [NASTY] Develop a MI-SOCP model of SFSP-DCR for the group-based approximation of SRP

$$\theta_{ij} = 3\frac{2^{\lceil \log_2 w_{ij} L / r_{ij} \rceil}}{w_{ij}} + 2\frac{L}{w_{ij}}$$

You are allowed to cheat, using the fact that

$$3\frac{L}{r_{ij}} + 2\frac{L}{w_{ij}} \leq \theta_{ij} \leq 6\frac{L}{r_{ij}} + 2\frac{L}{w_{ij}}$$

but it's not as fun

Homework – Algorithms

1. Construct an example where the improved ERA-I fails to find the correct solution (the shortest path of some step-length l is never the current path when d exits Q)
2. Prove that the pseudo-polynomial algorithm works
3. Prove that the FPTAS works

Homework – Algorithms

1. Construct an example where the improved ERA-I fails to find the correct solution (the shortest path of some step-length l is never the current path when d exits Q)
2. Prove that the pseudo-polynomial algorithm works
3. Prove that the FPTAS works
4. [NASTY] Develop Lagrange, then Benders' in details
5. [NASTY] Develop Benders', than Lagrange in details

References

Scheduling Algorithms, Network Calculus

Cruz "A calculus for network delay" (part I/part II), *IEEE Trans. on Information Theory*, 1991

Zhang "Service Disciplines for Guaranteed Performance Service in Packet-Switching Networks", *Proceedings of the IEEE*, 1995

Shreedhar, Varghese "Efficient Fair Queueing Using Deficit Round Robin", *IEEE Trans. on Networking*, 1996.

Stiliadis, Varma "Latency-Rate Servers: A General Model for Analysis of Traffic Scheduling Algorithms", *IEEE Trans. on Networking*, 1998

Second-Order Cone Programs

Ben Tal "Lecture Notes of the Course on Conic and Robust Optimization", 2002

References

Perspective stuff

F., Gentile "Perspective Cuts for a Class of Convex 0-1 Mixed Integer Programs", *Math. Prog.*, 2006

Günlük, Linderoth "Perspective reformulations of MINLPs with indicator variables", *Math. Prog.*, 2010

Hijazi, Bonami, Cornuejols, Ouorou "Mixed integer nonlinear programs featuring "On/Off" constraints: Convex analysis and applications" *Electronic Notes in Discrete Mathematics*, 2010

DCR

Orda "Routing with end-to-end QoS guarantees in broadband networks", *IEEE Trans. on Networking*, 1999

F., Galli, Scutellà "Delay-Constrained Shortest Paths: Approximation Algorithms and Second-Order Cone Models", *JOTA*, 2014

F., Galli, Stea "Optimal Joint Path Computation and Rate Allocation for Real-time Traffic", *The Computer Journal*, 2014