0-1 Reformulations of the Network Loading Problem

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Abstract

We study 0-1 reformulations of the Network Loading problem, a capacitated network design problem which is usually modeled with general integer variables to represent design decisions on the number of facilities to install on each arc of the network. The reformulations are based on the multiple choice model, a generic approach to represent piecewise linear costs using 0-1 variables. This model is improved by the addition of extended linking inequalities, derived from variable disaggregation techniques. We show that these extended linking inequalities for the 0-1 model are equivalent to the residual capacity inequalities, a class of valid inequalities derived for the model with general integer variables. This result yields three strategies to compute the same lower bound on the optimal value of the problem: 1) A Dantzig-Wolfe (DW) approach applied to the model with general integer variables; 2) A cutting-plane algorithm based on the residual capacity inequalities; 3) A *Structured DW* method that solves the 0-1 reformulation with extended linking inequalities by variables and constraints generation.

Keywords: Capacitated Network Design, Network Loading, Reformulation, Dantzig-Wolfe Decomposition

The Network Loading problem (NL) can be described as follows. Given a directed network $G = (N, A)$, where N is the set of nodes and A is the set of arcs, we must satisfy the communication demands between several origin-destination pairs, represented by the set of commodities K. For each commodity k, we denote by d^k the positive demand that must flow between the origin $O(k)$ and the destination $D(k)$. While flowing along an arc (i, j), a communication consumes some of the arc capacity; the capacity is originated by installing on some of the arcs any number of *facilities*. Installing one facility on arc (i, j) provides a positive capacity u_{ij} at a (nonnegative) cost f_{ij}; a nonnegative routing cost c_{ij}^k also has to be paid for each unit of commodity k moving through arc $(i, j) \in A$. The problem consists in minimizing the sum of all costs, while satisfying demand requirements and capacity constraints.

We define nonnegative *flow variables* x_{ij}^k, which represent the fraction of the flow of commodity k on arc $(i, j) \in A$, i.e., $d^k x_{ij}^k$ is the flow of commodity k on arc (i, j). We also introduce general integer *design variables* y_{ij}, which define the number of facilities to install on arc (i, j). The problem can then be formulated as follows:

$$\min \sum_{k\in K} \sum_{(i,j)\in A} d^k c_{ij}^k x_{ij}^k + \sum_{(i,j)\in A} f_{ij} y_{ij}, \tag{1}$$

$$\sum_{j\in N} x_{ij}^k - \sum_{j\in N} x_{ji}^k = \begin{cases} 1, & if\ i = O(k), \\ -1, & if\ i = D(k), \\ 0, & if\ i \neq O(k), D(k), \end{cases} \qquad \forall\ i \in N,\ k \in K, \tag{2}$$

$$\sum_{k\in K} d^k x_{ij}^k \leq u_{ij} y_{ij}, \quad \forall\ (i, j) \in A, \tag{3}$$

$$0 \leq x_{ij}^k \leq 1, \quad \forall\ (i, j) \in A,\ k \in K, \tag{4}$$

$$y_{ij} \geq 0, \quad \forall\ (i, j) \in A, \tag{5}$$

$$y_{ij}\ integer, \quad \forall\ (i, j) \in A. \tag{6}$$

We will denote this model as I.

Since $f_{ij} \geq 0$, we have $y_{ij} \leq \left\lceil \frac{\sum_{k\in K} d^k}{u_{ij}} \right\rceil = T_{ij}$ for each arc (i, j). It follows that the problem can be reformulated with $2 \sum_{(i,j)\in A} T_{ij}$ *auxiliary variables* using the sets $S_{ij} = \{1, \ldots, T_{ij}\}$:

$$y_{ij}^s = \begin{cases} 1, & if\ y_{ij} = s, \\ 0, & otherwise, \end{cases} \qquad \forall\ s \in S_{ij}, \tag{7}$$

$$x_{ij}^s = \begin{cases} \sum_{k\in K} d^k x_{ij}^k, & if\ y_{ij} = s, \\ 0, & otherwise, \end{cases} \qquad \forall\ s \in S_{ij}. \tag{8}$$

We can then write the problem as above, but with the additional constraints:

$$y_{ij} = \sum_{s \in S_{ij}} s y_{ij}^s, \quad \forall (i,j) \in A, \tag{9}$$

$$\sum_{k \in K} d^k x_{ij}^k = \sum_{s \in S_{ij}} x_{ij}^s, \quad \forall (i,j) \in A, \tag{10}$$

$$(s-1) u_{ij} y_{ij}^s \leq x_{ij}^s \leq s u_{ij} y_{ij}^s, \quad (i,j) \in A, s \in S_{ij}, \tag{11}$$

$$\sum_{s \in S_{ij}} y_{ij}^s \leq 1, \quad (i,j) \in A, \tag{12}$$

$$y_{ij}^s \geq 0, \quad (i,j) \in A, s \in S_{ij}, \tag{13}$$

$$y_{ij}^s \ integer, \quad (i,j) \in A, s \in S_{ij}. \tag{14}$$

We will denote this model as B.

Note that we can remove constraints (5) and (6), but also constraints (3), which are implied by (9) and (11). Consequently, we can project out variables y_{ij} and obtain a formulation expressed only in terms of the auxiliary binary variables y_{ij}^s, in addition to the flow variables. This formulation corresponds to the so-called *multiple choice* model [2], which can also be derived by interpreting the problem as a Multicommodity flow formulation with piecewise linear costs, each segment of the corresponding cost function on any given arc representing the number of facilities to install on this arc. Using this interpretation, one can also derive two other textbook formulations for piecewise linear cost function, the so-called incremental and convex combination models [2]. These three formulations are not only equivalent in terms of IP, but also as LP, and they all provide the same approximation, which corresponds to the lower convex envelope of the cost function. As in [3], we study the multiple choice model, as it lends itself nicely to the addition of simple valid inequalities derived from *variable disaggregation* techniques.

These techniques are based on the addition of the following *extended auxiliary variables*:

$$x_{ij}^{ks} = \begin{cases} x_{ij}^k, & if\ y_{ij} = s, \\ 0, & otherwise, \end{cases} \quad \forall\ s \in S_{ij}. \tag{15}$$

In terms of linear equations, these variables are defined as follows:

$$x_{ij}^k = \sum_{s \in S_{ij}} x_{ij}^{ks}, \quad \forall (i,j) \in A, k \in K, \tag{16}$$

$$x_{ij}^s = \sum_{k \in K} d^k x_{ij}^{ks}, \quad \forall (i,j) \in A, s \in S_{ij}. \tag{17}$$

The following *extended linking* constraints

$$x_{ij}^{ks} \leq y_{ij}^{s}, \quad \forall (i,j) \in A, k \in K, s \in S_{ij} \tag{18}$$

are then redundant in the IP, but not in the LP relaxation of model B; we will denote the resulting *extended* model as B^+.

We will now compare the different models and some of their relaxations. We will denote as $F(M)$, $conv(F(M))$ and $LP(M)$, the feasible set, its convex hull and the LP relaxation, respectively, of any given model M. In our analysis, we will consider the Lagrangian relaxation of the flow conservation equations (2). The resulting Lagrangian subproblem and Lagrangian dual for any model M will be denoted as $LS(M)$ and $LD(M)$, respectively. Finally, we will say that two models are equivalent if their optimal values are the same, for any values of the costs (we only require that all costs are nonnegative).

We first give without proofs a series of obvious equivalencies between the models.

Proposition 1 *I and B^+ are equivalent.*

Proposition 2 *$LS(I)$ and $LS(B^+)$ are equivalent.*

Proposition 3 *$LD(I)$ and $LD(B^+)$ are equivalent.*

Now, we turn our attention to the LP relaxations of the two formulations and compare them with the Lagrangian relaxation of the flow conservation equations.

Proposition 4 *$F(LP(LS(B^+))) = conv(F(LS(B^+)))$.*

Proof: See Croxton, Gendron, Magnanti (2004) [3].

From standard Lagrangian duality theory, we then immediately obtain the following result.

Proposition 5 *$LP(B^+)$ and $LD(B^+)$ are equivalent.*

With respect to formulation I, we know an explicit description of $conv(F(LS(I)))$ [4]. Indeed, is is sufficient to add the so-called *residual capacity* inequalities to $F(LP(LS(I)))$ to obtain the convex hull. Let I^+ be the model obtained by appending to I the residual capacity inequalities. The next two results follow immediately from the convex hull result of [4].

Proposition 6 *$F(LP(LS(I^+))) = conv(F(LS(I)))$.*

Proposition 7 *$LP(I^+)$ and $LD(I)$ are equivalent.*

The following Theorem is then an immediate consequence of Propositions 3, 5 and 7.

Theorem 8 *$LP(B^+)$ and $LP(I^+)$ are equivalent.*

Thus, the improvement in the LP relaxation bound provided by the extended formulation B^+ is equivalent to the addition of the residual capacity inequalities to formulation I.

It is interesting to contrast the three possible ways that we now have to obtain the same lower bound on the optimal value of the problem:

- Apply the Dantzig-Wolfe approach to the original model I, that is, solve – by variables generation – an LP with exponentially many variables and "few" constraints.
- Apply a cutting-plane algorithm [5, 1] to solve formulation $LP(I^+)$ (the separation problem for the residual capacity inequalities is solvable in polynomial time [1]), that is, solve – by constraints generation – an LP with exponentially many constraints and "few" variables.
- Apply a *Structured DW* approach to formulation $LP(B^+)$, that is, solve – by variables *and* constraints generation – an LP with a pseudo-polynomial number of both variables and constraints.

The details of implementing a Structured DW approach, as well as computational experiments comparing these three strategies will be reported at the conference.

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