Quasi-Separable Dantzig-Wolfe Reformulations for Network Design

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Abstract. Under mild assumptions that are satisfied for many network design models, we show that the Lagrangian dual obtained by relaxing the flow constraints is what we call "quasi-separable." This property implies that the Dantzig-Wolfe (DW) reformulation of the Lagrangian dual exhibits two sets of convex combination constraints, one in terms of the design variables and the other in terms of the flow variables, the latter being linked to the design variables. We compare the quasi-separable DW reformulation to the standard disaggregated DW reformulation. We illustrate the concepts on a particular case, the budget-constrained multicommodity capacitated unsplittable network design problem.

Keywords: Lagrangian relaxation, Dantzig-Wolfe reformulations, network design

1 Introduction

We consider a large class of network design models that can be represented by the following generic mixed-integer linear program (MILP), denoted (ND) [1]:

$$v(ND) = \min cx + fy \tag{1}$$
$$Ax = b \tag{2}$$
$$Dx + Ey \geq g \tag{3}$$
$$Hy \geq p \tag{4}$$
$$x \in \mathcal{X} \subset \mathbb{R}^n_+ \tag{5}$$
$$y \in \mathcal{Y} \subset \mathbb{Z}^m_+ \tag{6}$$

where $v(M)$ denotes the optimal value of any model (M) and the rational vectors b, c, f, g, p and rational matrices A, D, E, H have appropriate dimensions. The sets $\mathcal{X}$ and $\mathcal{Y}$ are the domains of the *flow variables* x and the *design variables* y, respectively. We assume the two sets are bounded and defined by three types of constraints: integer-valued bounds on individual variables; simplex constraints on subsets of the variables, which may be included in the definition of $\mathcal{Y}$, but not in that of $\mathcal{X}$; integrality constraints, which are included in the definition of $\mathcal{Y}$, but not necessarily in that of $\mathcal{X}$. When integrality constraints are relaxed, we denote the corresponding domains of variables $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$.

We call (2) the *flow constraints*, (3) the *linking constraints* and (4) the *design constraints*. To analyse these constraints, we introduce the corresponding sets $\mathcal{Q}_F = \{ (x, y) \in \mathcal{X} \times \mathcal{Y} \,|\, Ax = b \}$, $\mathcal{Q}_L = \{ (x, y) \in \mathcal{X} \times \mathcal{Y} \,|\, Dx + Ey \geq g \}$, and $\mathcal{Q}_D = \{ (x, y) \in \mathcal{X} \times \mathcal{Y} \,|\, Hy \geq p \}$. The associated linear programming (LP) relaxation polyhedra, obtained by substituting $\mathcal{X}$ with $\overline{\mathcal{X}}$ and $\mathcal{Y}$ with $\overline{\mathcal{Y}}$, will be denoted respectively with $\mathcal{P}_F$, $\mathcal{P}_L$, and $\mathcal{P}_D$. Note that sets $\mathcal{Q}_F$ and $\mathcal{Q}_D$ ($\mathcal{P}_F$ and $\mathcal{P}_D$) are defined on all the space, but in fact only concern a subset of the variables; we will therefore denote by $\mathcal{Q}_F^x \subset \mathbb{R}_+^n$ and $\mathcal{Q}_D^y \subset \mathbb{Z}_+^m$ (and similarly for the continuous relaxation) their projection on the set of relevant variables.

Many solution methods for network design models that can be cast as special cases of (ND) rely on *Lagrangian relaxation* strategies. These consist in relaxing, in a Lagrangian way, either the linking constraints (3) [2,3] or the flow constraints (2) [2,4,5,6,7]. These strategies give rise to two Lagrangian dual programs: the *linking relaxation dual*, noted (LD_L), and the *flow relaxation dual*, noted (LD_F), respectively.

In (LD_L), relaxing the linking constraints (3) yields a Lagrangian relaxation that can be decomposed into two independent subproblems: one in the x variables, and one in the y ones. Hence, the bound strength of (LD_L) can be estimated using the primal interpretation of Lagrangian duality: the general result of [8] reads

$$
\begin{aligned}
v(LD_L) &= \min\{ cx + fy \,|\, (x, y) \in \mathcal{P}_L \cap conv(\mathcal{Q}_F \cap \mathcal{Q}_D) \} \\
&= \min\{ cx + fy \,|\, (x, y) \in \mathcal{P}_L \cap conv(\mathcal{Q}_F) \cap conv(\mathcal{Q}_D) \} \\
&= \min\{ cx + fy \,|\, (x, y) \in \mathcal{P}_L \,,\, x \in conv(\mathcal{Q}_F^x) \,,\, y \in conv(\mathcal{Q}_D^y) \} \;\;,
\end{aligned}
$$

where $conv(\mathcal{C})$ denotes the convex hull of the set $\mathcal{C}$. Since the Lagrangian relaxation is separable in two independent problems, we can write down its *Dantzig-Wolfe (DW) reformulation* using two sets of convex combination constraints. That is, being $\{ x^s \}_{s \in \mathcal{S}}$ and $\{ y^t \}_{t \in \mathcal{T}}$ the sets of extreme points of $conv(\mathcal{Q}_F^x)$ and $conv(\mathcal{Q}_D^y)$, respectively, one has the following explicit form

$$
\begin{aligned}
v(LD_L) = \min\; & cx + fy && (1)\\
& Dx + Ey \geq g && (3)\\
& x = \textstyle\sum_{s \in \mathcal{S}} \lambda^s x^s \;\;,\;\; \sum_{s \in \mathcal{S}} \lambda^s = 1 \;\;,\;\; \lambda \geq 0 && (7)\\
& y = \textstyle\sum_{t \in \mathcal{T}} \gamma^t y^t \;\;,\;\; \sum_{t \in \mathcal{T}} \gamma^t = 1 \;\;,\;\; \gamma \geq 0 && (8)
\end{aligned}
$$

The second strategy, that of relaxing the flow constraints (2), does not yield a separable Lagrangian relaxation. That is, for the corresponding Lagrangian dual (LD_F) one has

$$
v(LD_F) = \min\{ cx + fy \,|\, (x, y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L \cap \mathcal{Q}_D) \}
$$

and the relevant set is $\{(x^r, y^r)\}_{r\in\mathcal{R}}$, containing the extreme points of $conv(\mathcal{Q}_L \cap \mathcal{Q}_D)$, which yields the DW reformulation

$$v(LD_F) = \min cx + fy \tag{1}$$
$$Ax = b \tag{2}$$
$$(x, y) = \textstyle\sum_{r\in\mathcal{R}} \theta^r (x^r, y^r) \;\;,\;\; \sum_{r\in\mathcal{R}} \theta^r = 1 \;\;,\;\; \theta \geq 0 \tag{9}$$

In Section 2, we show that, under mild assumptions that are satisfied for many network design models, (LD_F) can be reformulated in an "almost" separable form, which we call *quasi-separable*. Then, in Section 3 we compare the quasi-separable DW reformulation to a disaggregated DW reformulation, which we define. Finally, in Section 4 we illustrate our results on a special case of (ND), the *budget-constrained multicommodity capacitated unsplittable network design problem (BMCUND)*.

2 Quasi-Separable Lagrangian Dual

To derive the quasi-separable DW reformulation of (LD_F), we recall recently published results [1] concerning the Lagrangian subproblem associated with the relaxation of the flow constraints. If we denote by π the (unrestricted) Lagrange multipliers associated with the flow constraints (2), and with $\overline{c} = c - \pi$, the Lagrangian relaxation can be written as

$$v(LR_F^{\overline{c}}) = \min \left\{ \overline{c}x + fy \,|\, (x, y) \in \mathcal{X} \times \mathcal{Y} \;,\; Dx + Ey \geq g \;,\; Hy \geq p \right\} \;\;.$$

Using a Benders' decomposition strategy, we consider y as "complicating" variables and define the Benders subproblem

$$v(BS^{\overline{c}}(y)) = \min\{\, \overline{c}x \,|\, x \in \mathcal{Q}_L(y) \,\}$$

where $\mathcal{Q}_L(y) = \{\, x \in \mathcal{X} \,|\, Dx \geq g - Ey \,\}$; hence, the Lagrangian relaxation can be rewritten as

$$v(LR_F^{\overline{c}}) = \min \left\{ \, fy + v(BS^{\overline{c}}(y)) \,|\, y \in \mathcal{Q}_D^y \,\right\} \;\;.$$

The following assumption holds for many network design models where y are binary variables.

Assumption QS. *[1] $v(BS^{\overline{c}}(y))$ can be written as a linear function of $y \in \mathcal{Y}$: for any cost vector $\overline{c}$, there exists a cost vector $w^{\overline{c}}$ such that $v(BS^{\overline{c}}(y)) = w^{\overline{c}}y$.*

The following proposition, due to [1], shows that Assumption QS allows to decompose the Lagrangian relaxation by optimizing first over $x \in \mathcal{Q}_L(y)$, then over $y \in \mathcal{Q}_D^y$, giving rise to a quasi-separable Lagrangian dual.

Proposition 1. *[1] Under Assumption QS, it holds*

$$conv(\mathcal{Q}_L \cap \mathcal{Q}_D) = conv(\mathcal{Q}_L) \cap conv(\mathcal{Q}_D) \;\;.$$

Proof. We prove that, however chosen cost vectors $\bar{c}$ and $\bar{f}$, minimizing them over the two sets yields the same result. In fact

$$
\begin{aligned}
\min_{(x,y)\in conv(\mathcal{Q}_L\cap\mathcal{Q}_D)} \bar{c}x+\bar{f}y = \min_{(x,y)\in\mathcal{Q}_L\cap\mathcal{Q}_D} \bar{c}x+\bar{f}y &= \min_{y\in\mathcal{Q}_D^y} \bar{f}y + v(BS^{\bar{c}}(y)) \\
&= \min_{y\in\mathcal{Q}_D^y} (\bar{f}+w^{\bar{c}})y \qquad = \min_{y\in conv(\mathcal{Q}_D^y)} (\bar{f}+w^{\bar{c}})y \\
&= \min_{y\in conv(\mathcal{Q}_D^y)} \Big(\bar{f}y + \min_{x\in\mathcal{Q}_L(y)} \bar{c}x\Big) \\
&= \min_{y\in conv(\mathcal{Q}_D^y)} \Big(\bar{f}y + \min_{x\in conv(\mathcal{Q}_L(y))} \bar{c}x\Big) \\
&= \min_{y\in conv(\mathcal{Q}_D^y)} \Big(\min_{(x,y)\in conv(\mathcal{Q}_L)} \bar{c}x+\bar{f}y\Big) \\
&= \min_{(x,y)\in conv(\mathcal{Q}_L)\cap conv(\mathcal{Q}_D)} \bar{c}x+\bar{f}y \quad .
\end{aligned}
$$

Corollary 1. *Under Assumption QS, it holds*

$$
v(LD_F) = \min_{y\in conv(\mathcal{Q}_D^y)} \Big(fy + \min_{x\in\mathcal{P}_F^x\cap conv(\mathcal{Q}_L(y))} cx\Big) \quad .
$$

The next proposition gives sufficient conditions for a model of the form (ND) to satisfy Assumption QS.

Proposition 2. *Consider any model (ND) such that, for some set J, both $\mathcal{Y}$ and $\mathcal{X}$ decompose over J, i.e., $\mathcal{Y} = \times_{j\in J}\mathcal{Y}^j$ and $\mathcal{X} = \times_{j\in J}\mathcal{X}^j$. Thus, $x = [\,x^j\,]_{j\in J}$ and $y = [\,y^j\,]_{j\in J}$. Let $I(j)$ be the set of indices of the variables in y^j, i.e., $y^j = [\,y_i\,]_{i\in I(j)}$, and $I = \cup_{j\in J} I(i)$. If*

1. *$\mathcal{Y}^j = \{\, y^j \in \{\,0\,,\,1\,\}^{|I(j)|} \mid \sum_{i\in I(j)} y_i \le 1\,\}$;*
2. *$\mathcal{Q}_L(y)$ decomposes over J: $\mathcal{Q}_L(y) = \times_{j\in J}\{\, x^j \in \mathcal{X}^j \mid D^j x^j \le E^j y^j\,\}$ for rational matrices $D^j \ge 0$ and $E^j \ge 0$ of appropriate dimensions;*

then model (ND) satisfies Assumption QS.

Proof. Under the assumptions, we can write the Benders subproblem as

$$
v(BS^{\bar{c}}(y)) = \textstyle\sum_{j\in J} \min\big\{\, \bar{c}_j x^j \mid x^j \in \mathcal{X}^j \;,\; D^j x^j \le E^j y^j \,\big\} \quad .
$$

For each $j \in J$, if $y^j = 0$ then the unique solution is $x^j = 0$. Otherwise, let $i \in I(j)$ be unique index of the nonzero variable in y^j: then, x^j can be obtained by solving

$$
w_i^{\bar{c}} = \min\big\{\, \bar{c}_j x^j \mid x^j \in \mathcal{X}^j \;,\; D^j x^j \le e_i \,\big\}
$$

where e_i is the column of E^j corresponding to y_i. Note that this problem is feasible, as $x^j = 0$ is a solution, and bounded, since $\mathcal{X}^j$ is bounded. Thus, $v(BS^{\bar{c}}(y)) = \sum_{i\in I} w_i^{\bar{c}} y_i$, and Assumption QS is satisfied. □

Using the same notation as in the proof of Proposition 2, we define for each $i \in I$

$$\mathcal{Q}_L^x(i) = \{\, x^j \in \mathcal{X}^j \mid D^j x^j \leq e_i \,\}$$

where j is the unique index such that $i \in I(j)$ (this should be denoted by "$j(i)$", but we will avoid it whenever i is clear from the context, as we will use y_i^j for y_i only if necessary). We then denote as $\{\, x^{j,s} \,\}_{s \in \mathcal{S}(i)}$ the set of extreme points of $conv(\mathcal{Q}_L^x(i))$.

Proposition 3. *Under the assumptions of Proposition 2, we have*

$$conv(\mathcal{Q}_L(y)) = \bigtimes_{j \in J} \left\{ x^j \in \mathcal{X}^j \,\middle|\, \begin{array}{ll} x^j = \sum_{i \in I(j)} \sum_{s \in \mathcal{S}(i)} \omega^s x^{j,s} & \\ y_i = \sum_{s \in \mathcal{S}(i)} \omega^s & i \in I(j) \\ \omega^s \geq 0 & i \in I(j) \;,\; s \in \mathcal{S}(i) \end{array} \right\}$$

Proof. Due to the assumptions we have

$$conv(\mathcal{Q}_L(y)) = \bigtimes_{j \in J} \{\, \{\, 0 \,\} \cup \bigcup_{i \in I(j)} conv(\mathcal{Q}_L^x(i)) \,\} \;\; .$$

Clearly, we only need to discuss each $j \in J$ (with the corresponding y^j and x^j) separately. Consider any

$$x^j = \sum_{i \in I(j)} \sum_{s \in \mathcal{S}(i)} \omega^s x^{j,s} \;\; .$$

If $y^j = 0$, then $x^j = 0$. Otherwise, let i be the unique index in $I(j)$ such that $y_i = 1$. Clearly, $y_h = 0$ for $h \in I(j) \setminus \{\, i \,\}$; therefore, $\omega^s = 0$ for all $s \in \mathcal{S}(h)$. Consequently

$$\sum_{s \in \mathcal{S}(i)} \omega^s x^{j,s} = x^j \in conv(\mathcal{Q}_L^x(i))$$

which implies the result. □

With the same notation as in Section 1, we can now write the quasi-separable DW reformulation of the flow relaxation dual:

$$\begin{aligned} v(LD_F) = \min\; & cx + fy && (1) \\ & Ax = b && (2) \\ & y = \textstyle\sum_{t \in \mathcal{T}} \gamma^t y^t \;\;,\;\; \sum_{t \in \mathcal{T}} \gamma^t = 1 \;\;,\;\; \gamma \geq 0 && (8) \\ & x^j = \textstyle\sum_{i \in I(j)} \sum_{s \in \mathcal{S}(i)} \omega^s x^{j,s} \qquad j \in J && (10) \\ & y_i = \textstyle\sum_{s \in \mathcal{S}(i)} \omega^s \qquad i \in I && (11) \\ & \omega^s \geq 0 \qquad i \in I \;,\; s \in \mathcal{S}(i) && (12) \end{aligned}$$

This DW reformulation corresponds to the expression of (LD_F) given by Corollary 1. Indeed, constraints (8) correspond to $y \in conv(\mathcal{Q}_D^y)$, constraints (2) correspond to $x \in \mathcal{P}_F^x$, and, by Proposition 3, constraints (10)–(12) correspond to $x \in conv(\mathcal{Q}_L^x(y))$. The quasi-separable DW reformulation relies on the fact that the Benders subproblem derived from the Lagrangian subproblem decomposes by $j \in J$. As such, it bears close resemblance to a disaggregated DW reformulation that could be derived from the Lagrangian relaxation of constraints (2) and (4). Next, we compare these two reformulations, showing that they are essentially the same when $\mathcal{P}_D$ is an integral polyhedron.

3 Comparison with Disaggregated DW Reformulation

The disaggregated DW reformulation is obtained by relaxing in a Lagrangian way both the flow constraints (2) and the design constraints (4). The resulting Lagrangian subproblem decomposes by $j \in J$, i.e., its feasible domain is $\bigtimes_{j \in J} \mathcal{Q}_L(j)$, where

$$\mathcal{Q}_L(j) = \{ (x^j, y^j) \in \mathcal{X}^j \times \mathcal{Y}^j \mid D^j x^j \leq E^j y^j \} \ .$$

The disaggregated DW reformulation of the corresponding Lagrangian dual, called the *flow-design relaxation dual* and denoted (LD_{FD}), can then be written as follows, where $\{ (x^{j,r} \, , \, y^{j,r}) \}_{r \in \mathcal{R}^j}$ are the extreme points of $conv(\mathcal{Q}_L(j))$ excluding $(0 \, , \, 0)$:

$$\begin{aligned} v(LD_{FD}) = \min \ & cx + fy && && (1)\\ & Ax = b && && (2)\\ & Hy \geq p && && (4)\\ & x^j = \textstyle\sum_{r \in \mathcal{R}^j} \theta^{j,r} x^{j,r} && j \in J && (13)\\ & y^j = \textstyle\sum_{r \in \mathcal{R}^j} \theta^{j,r} y^{j,r} && j \in J && (14)\\ & \textstyle\sum_{r \in \mathcal{R}^j} \theta^{j,r} \leq 1 \ \ , \ \ \theta^j \geq 0 && j \in J && (15) \end{aligned}$$

Note that, for any $j \in J$, there is a one-to-one correspondence between the extreme points $\{ (x^{j,r} \, , \, y^{j,r}) \}_{r \in \mathcal{R}^j}$ of $conv(\mathcal{Q}_L(j)) \setminus \{ (0 \, , \, 0) \}$ and the extreme points $\{ x^{j,s} \}_{s \in \mathcal{S}(i)}$ of $conv(\mathcal{Q}^x_L(i))$ for some $i \in I(j)$. Indeed, for each $s \in \cup_{i \in I(j)} \mathcal{S}(i)$, there exists a unique $r \in \mathcal{R}^j$ such that $x^{j,r} = x^{j,s}$ and $y_i = 1$. We denote as $r = r(s)$ this unique index.

In general, we have $v(LD_F) \geq v(LD_{FD})$ and the inequality can be strict if $conv(\mathcal{Q}_D) \subset \mathcal{P}_D$. However, if $\mathcal{P}_D$ is an integral polyhedron, then $v(LD_F) = v(LD_{FD})$. In fact, the next proposition show that when $conv(\mathcal{Q}_D) = \mathcal{P}_D$ the quasi-separable and disaggregated DW reformulations are essentially identical.

Proposition 4. *If $\mathcal{P}_D$ is an integral polyhedron, then there is a one-to-one correspondence between the solutions of the quasi-separable and disaggregated DW reformulations, given by*

$$\theta^{j,r} y^{j,r}_i = \omega^s \qquad for \qquad j \in J \ , \ i \in I(j) \ , \ s \in \mathcal{S}(i) \ , \ r = r(s) \ . \qquad (16)$$

Proof. The objective functions and the flow constraints are the same in the two models. In addition, since $\mathcal{P}_D$ is an integral polyhedron, (8) are equivalent to (4). Because of (16), nonnegativity constraints (12) and (15) are equivalent. For any $j \in J$, the identity $\sum_{i \in I(j)} y^{j,r}_i = 1$ is true for all $r \in \mathcal{R}^j$, hence:

1. (10) and (13) are identical, since

$$x^j = \textstyle\sum_{r \in \mathcal{R}^j} \theta^{j,r} x^{j,r} = \sum_{i \in I(j)} \sum_{r \in \mathcal{R}^j} \theta^{j,r} y^{j,r}_i x^{j,r} = \sum_{i \in I(j)} \sum_{s \in \mathcal{S}(i)} \omega^s x^{j,s} \ .$$

2. (11) and (14) are identical, since $y_i = \sum_{r \in \mathcal{R}^j} \theta^{j,r} y^{j,r}_i = \sum_{s \in \mathcal{S}(i)} \omega^s$.

3. (15) is implied by (11) and the definition of $\mathcal{Y}$:

$$\sum_{r\in\mathcal{R}^j}\theta^{j,r} = \sum_{i\in I(j)}\sum_{r\in\mathcal{R}^j}\theta^{j,r}y_i^{j,r} = \sum_{i\in I(j)}\sum_{s\in\mathcal{S}(i)}\omega^s = \sum_{i\in I(j)} y_i \le 1 \ .$$

This concludes the proof. □

This proposition implies that the quasi-separable DW reformulation really brings something more than the disaggregated DW reformulation only for problems where $\mathcal{P}_D$ is not an integral polyhedron. In the next section we present such a problem.

4 Illustration with the BMCUND

The Budget-Constrained Multicommodity Capacitated Unsplittable Network Design problem (BMCUND) is defined on a directed graph $G = (N, J)$, where N is the set of nodes and J is the set of arcs. For each node $n \in N$ we define the sets of outgoing and incoming arcs, J_n^+ and J_n^-, respectively. Each commodity $k \in K$ corresponds to an origin–destination pair such that d_k units of flow must travel between the origin $O(k)$ and the destination $D(k)$ using a single path; this is why the problem is termed *unsplittable*, to distinguish it from the splittable variant where the flow of each commodity can be split among several paths. There is a limited budget M on the global investment costs to select the arcs to be used, where using arc $j \in J$ incurs a fixed cost $f^j \ge 0$ and provides a capacity $u^j > 0$. The objective function to be minimized are the routing costs $c_k^j \ge 0$ for each unit of commodity $k \in K$ through arc $j \in J$. We introduce two sets of variables to model the problem: x_k^j is 1 if the demand d_k of commodity k flows on arc j, and 0, otherwise; y^j is 1, if arc j is used, and 0, otherwise. The model is then written as follows:

$$\begin{aligned}
v(BND) = \min\ & \textstyle\sum_{j\in J}\sum_{k\in K} d_k c_k^j x_k^j && && (17)\\
& \textstyle\sum_{j\in J_n^+} x_k^j - \sum_{j\in J_n^-} x_k^j = b_k^n && n\in N \ ,\ k\in K && (18)\\
& \textstyle\sum_{k\in K} d_k x_k^j \le u^j y^j && j\in J && (19)\\
& x_k^j \le y^j && j\in J \ ,\ k\in K && (20)\\
& \textstyle\sum_{j\in J} f^j y^j \le M && && (21)\\
& x_k^j \in \{\,0\,,\,1\,\} && j\in J \ ,\ k\in K && (22)\\
& y^j \in \{\,0\,,\,1\,\} && j\in J && (23)
\end{aligned}$$

where b_k^n is the supply of node n for commodity k, i.e., 1 for $n = O(k)$, -1 for $n = D(k)$, and 0 otherwise. This model is a special case of (ND) for which the sets are defined as $\mathcal{X} = \{\, x = [\, x_k^j \,]_{j\in J, k\in K} \,|\, (22)\,\}$, $\mathcal{Y} = \{\, y = [\, y^j \,]_{j\in J} \,|\, (23)\,\}$, $\mathcal{Q}_F^x = \{\, x \in \mathcal{X} \,|\, (18)\,\}$, $\mathcal{Q}_L = \{\, (x, y) \in \mathcal{X}\times\mathcal{Y} \,|\, (19)\ ,\ (20)\,\}$, $\mathcal{Q}_D^y = \{\, y \in \mathcal{Y} \,|\, (21)\}$, and $I = J$. Relaxing the flow constraints in a Lagrangian way yields

$$\min\left\{\ \textstyle\sum_{j\in J}\sum_{k\in K} \bar{c}_k^j x_k^j \,|\, (x, y) \in \mathcal{Q}_L \cap \mathcal{Q}_D \ \right\} \ ,$$

where $\overline{c}_k^j = d_k c_k^j + \pi_k^{t(j)} - \pi_k^{h(j)}$, π_k^n are the Lagrange multipliers, and $h(j)$ and $t(j)$ are the head and the tail of arc j, respectively. The Benders subproblem $(BS^{\overline{c}}(y))$ decomposes by arcs: for each $j \in J$, if $y^j = 0$, we obtain the trivial solution where all variables take value 0. If $y^j = 1$ instead, a 0–1 knapsack subproblem must be solved. Let $\widetilde{x}^j$ be the solution, with optimal value $w_j^{\overline{c}}$: if $w_j^{\overline{c}} < 0$, then $(\,\widetilde{x}^j\,,\,1\,)$ is the optimal solution, otherwise the all-0 solution is optimal. This shows that

$$v(BS^{\overline{c}}(y)) = \min\{\, \overline{c}x \mid x \in \mathcal{Q}_L^x(y) \,\} = \textstyle\sum_{j \in J} w_j^{\overline{c}} y^j = w^{\overline{c}} y \;\;,$$

i.e., Assumption QS is satisfied. Note that the assumption is also verified for the splittable version of the problem, which is analogous save that a continuous (rather than a 0–1) knapsack problem must be solved to compute $w_j^{\overline{c}}$.

Before presenting the DW reformulations of (LD_F), we note that the flow relaxation dual dominates both the linking relaxation dual (LD_L) and the flow-design relaxation dual (LD_{FD}). Indeed, $\mathcal{P}_F$ is an integral polyhedron, which implies that

$$\begin{aligned} v(LD_L) &= \min\{\, cx \mid (x,y) \in conv(\mathcal{Q}_F) \cap \mathcal{P}_L \cap conv(\mathcal{Q}_D) \,\} \\ &= \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap \mathcal{P}_L \cap conv(\mathcal{Q}_D) \,\} \\ &\leq \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L) \cap conv(\mathcal{Q}_D) \,\} \\ &= \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L \cap \mathcal{Q}_D) \,\} = v(LD_F) \;\; . \end{aligned}$$

The inequality can be strict if $conv(\mathcal{Q}_L) \subset \mathcal{P}_L$, which is possible since $\mathcal{P}_L$ is not an integral polyhedron, as the Benders subproblem reduces to a 0–1 knapsack problem. Also, since $\mathcal{P}_D$ is not an integral polyhedron, we have

$$\begin{aligned} v(LD_{FD}) &= \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L) \cap \mathcal{P}_D \,\} \\ &\leq \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L) \cap conv(\mathcal{Q}_D) \,\} \\ &= \min\{\, cx \mid (x,y) \in \mathcal{P}_F \cap conv(\mathcal{Q}_L \cap \mathcal{Q}_D) \,\} = v(LD_F) \;\; . \end{aligned}$$

The inequality can be strict if $conv(\mathcal{Q}_D) \subset \mathcal{P}_D$, which is possible since $\mathcal{Q}_D$ is a 0–1 knapsack set.

We now present and contrast the two DW reformulations of LD_F. With the notation set forth in Section 1, the standard DW reformulation is:

$$v(LD_F) = \min \; \textstyle\sum_{j \in J} \sum_{k \in K} d_k c_k^j x_k^j \tag{17}$$

$$\textstyle\sum_{j \in J_n^+} x_k^j - \sum_{j \in J_n^-} x_k^j = b_k^n \qquad n \in N \;\;,\;\; k \in K \tag{18}$$

$$x_k^j = \textstyle\sum_{r \in \mathcal{R}} \theta^r x_k^{j,r} \qquad j \in J \;\;,\;\; k \in K \tag{24}$$

$$y^j = \textstyle\sum_{r \in \mathcal{R}} \theta^r y^{j,r} \qquad j \in J \tag{25}$$

$$\textstyle\sum_{r \in \mathcal{R}} \theta^r = 1 \;\;,\;\; \theta \geq 0 \tag{26}$$

It is interesting to note that constraints (25) are redundant and can be removed. Indeed, any link between the flow and design variables is captured in the Lagrangian subproblem, since the design variables do not appear in the objective function of the BMCUND.

To derive the quasi-separable DW reformulation, we use the same notation set forth in Section 2; note that, in this case, $\mathcal{Q}^x_L(i) = \{\, x^j \in \mathcal{X}^j \mid \sum_{k \in K} d_k x^j_k \leq u^j \,\}$ and $i = j$, since each of the simplices in the general treatment is actually a single variable. Then,

$$
\begin{aligned}
v(LD_F) = \min\ & \textstyle\sum_{j \in J} \sum_{k \in K} d_k c^j_k x^j_k && && (17)\\
& \textstyle\sum_{j \in J^+_n} x^j_k - \sum_{j \in J^-_n} x^j_k = b^n_k && n \in N \;\;,\;\; k \in K && (18)\\
& y^j = \textstyle\sum_{t \in \mathcal{T}} \gamma^t y^{j,t} && j \in J && (27)\\
& \textstyle\sum_{t \in \mathcal{T}} \gamma^t = 1 \;\;,\;\; \gamma \geq 0 && && (8)\\
& x^j_k = \textstyle\sum_{s \in \mathcal{S}(j)} \omega^s x^{j,s}_k && j \in J \;\;,\;\; k \in K && (28)\\
& y^j = \textstyle\sum_{s \in \mathcal{S}(j)} \omega^s && j \in J && (29)\\
& \omega^s \geq 0 && j \in J \;,\; s \in \mathcal{S}(j) && (30)
\end{aligned}
$$

Note that (28) is somehow simpler than the general (10), again due to the fact that $I(j) = \{\, j \,\}$. Compared to the standard DW reformulation, the quasi-separable DW reformulation is larger, but has an obvious advantage when applying column generation: the same extreme point y^t of $conv(\mathcal{Q}^y_D)$ can be recombined with different corresponding extreme points of $conv(\mathcal{Q}^x_L(y^t))$, while many more columns with the same y^t, but with different x values, might be needed to solve the standard DW reformulation. This should result in much less column generation iterations when solving the quasi-separable DW reformulation, as already shown for disaggregated DW reformulations (e.g., [9]). Computational results on large-scale instances of the BMCUND will soon be reported to verify this assertion.

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