OPERATIONS RESEARCH
Vol. 54, No. 4, July–August 2006, pp. 767–775
ISSN 0030-364X | EISSN 1526-5463 | 06 | 5404 | 0767

informs

DOI 10.1287/opre.1060.0309

Solving Nonlinear Single-Unit Commitment Problems with Ramping Constraints

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We present a dynamic programming algorithm for solving the single-unit commitment (1UC) problem with ramping constraints and arbitrary convex cost functions. The algorithm is based on a new approach for efficiently solving the single-unit economic dispatch (ED) problem with ramping constraints and arbitrary convex cost functions, improving on previously known ones that were limited to piecewise-linear functions. For simple convex functions, such as the quadratic ones typically used in applications, the solution cost of all the involved (ED) problems, consisting of finding an optimal primal and dual solution, is $O(n^3)$. Coupled with a special visit of the state-space graph in the dynamic programming algorithm, this approach enables one to solve (1UC) with simple convex functions in $O(n^3)$ overall.

Subject classifications: dynamic programming; unit commitment problem; ramping constraints.
Area of review: Optimization.
History: Received July 2004; revision received March 2005; accepted May 2005.

1. Introduction

The single-unit commitment (1UC) problem requires operating one generating thermal unit optimally within a certain discretized time horizon. The cost (or revenue) for generating power varies with each time instant. The generating unit is subject to some technical restrictions, most notably *minimum up- and downtime* constraints, as well as upper and lower bounds over the produced power when the unit is operational.

(1UC) is a mixed-integer nonlinear problem, hence, in general nontrivial to solve. It is particularly relevant because it appears as a subproblem to be repeatedly solved within Lagrangian schemes for (multi-)unit commitment (UC) problems, which require coordination of the operations of several generating units, within a certain discretized time horizon, to satisfy a given power demand at minimum cost. These Lagrangian schemes are among the most efficient solution techniques for this class of difficult, large-scale mixed-integer nonlinear problems; see, e.g., Bacaud et al. (2001), Belloni et al. (2003), Borghetti et al. (2001), Borghetti et al. (2003a), Madrigal and Quintana (1999), Zhuang and Galiana (1988) among others, not least because they are easily extended to accommodate contributions from other types of generating units, such as hydroelectric ones. Also, Lagrangian techniques can be relatively easily extended to consider constraints arising from selling the generated power on a free market as in Borghetti et al. (2003b).

Within a Lagrangian approach, one (1UC) per each generating unit is repeatedly solved with varying objective function, whence the need for efficient solution methods for this problem. When no ramping constraints are imposed, (1UC) can be solved by means of a two-stage process: First, the optimal generated power, if the unit is committed, is independently computed for each time period, and then the optimal set of time periods where the unit has to be committed is computed, taking into account the results of the previous phase, by means of a simple dynamic programming procedure. The resulting algorithm has a complexity of $O(n)$, n being the number of time instants in the discretized time horizon if the start-up cost of the unit is *time invariant*, and $O(n^2)$ if the start-up cost of the unit is *time dependent*, i.e., the cost of committing the unit at a certain time instant depends on how long the unit has been uncommitted.

Unfortunately, this approach fails when ramping constraints need to be considered. Ramping constraints limit the maximum increase or decrease of generated power from one time instant to the next, and reflect the thermal and mechanical inertia that has to be overcome for the unit to increase or decrease its output. These phenomena cannot be disregarded for large units or if the time discretization interval is small (e.g., 15 minutes). The reason for the failure is that the variables representing the power output are no longer independent once commitment decisions have been made; rather, they are linked by the ramping constraints. Hence, it is no longer possible to determine the optimal generated power if the unit is committed independently for each time period. Thus, the dynamic programming procedure for the case without ramping constraints

cannot be extended to determine the optimal commitment. Discretizing the power variables space, one may keep using a standard dynamic programming procedure (Bechert and Kwatny 1972), but the computational burden increases considerably, and the obtained solution is an approximated one.

In Fan et al. (2002), an approach is presented for efficiently solving (1UC) with ramping constraints when the cost function is piecewise linear. The approach is based on the following idea: redefine the state space of the dynamic programming procedure so that computation of the state costs reduces to a convex (although harder than in the standard case) problem, the *economic dispatch with ramping constraints* (ED). The efficiency is obtained by using a constructive dynamic programming procedure that solves (ED) with a piecewise-linear cost function, similar to that of Bannister and Kaye (1991) and Travers and Kaye (1998). Thus, two nested dynamic programming procedures are employed to obtain an overall efficient approach.

However, in most cases the cost function of the real unit is modeled, in (1UC), with a quadratic function. Closely approximating the quadratic function with a piecewise-linear one may require a large number of pieces, thereby increasing the cost of the overall solution procedure. We propose an efficient algorithm for (ED) with general convex cost functions that solves all the $O(n^2)$ (ED)s required to perform the dynamic programming procedure on the commitment decisions in $O(n^3)$ in the case of quadratic cost functions. The algorithm is simple to implement and works for a very general form of (1UC) with time-varying upper and lower limits over the generated power, as well as time-varying and different limits for ramp-up and ramp-down constraints. Coupled with a special visit of the state-space graph in the dynamic programming algorithm, this enables one to solve (1UC) in $O(n^3)$ overall in the case of quadratic cost functions.

The structure of this paper is as follows. In §2, a formulation of (1UC) is briefly presented. In §3, the dynamic programming procedure, similar to that of Fan et al. (2002) for solving (1UC) is recalled, the corresponding (ED) problems are discussed, and the special visit is described that allows one to solve the problem in $O(n^3)$ once all the node costs have been computed. Then, in §4, the algorithm for solving (ED) is presented and analyzed. Finally, in §5, some computational results, obtained in the context of a Lagrangian approach to (UC), are presented for showing the efficiency of the proposed approach, and in §6 conclusions are drawn.

2. Formulation

The single-unit commitment problem (1UC) is as follows. A thermal generating unit burning some type of fuel (oil, gas, coal, ...) is given. The unit is characterized by a maximum and minimum power output, l^t and u^t, respectively, for each time instant (e.g., hour or half-hour) in a set $T = \{1, \ldots, n\}$, covering some time horizon (e.g., a day or a week). If the unit is committed (actively generating power) at time instant t, it is subject to a convex power-generating cost function $f^t(p_t)$, where p_t is the amount of power produced. In the following we will only assume that f^t is closed convex and that $f^t(0) = 0$ (any constant term in f^t can be associated with commitment variables, as discussed next). The operation of the unit must satisfy a number of technical constraints, typically the *minimum up- and downtime* ones: Whenever the unit is turned on it must remain committed for at least τ^+ consecutive time instants, and, analogously, whenever the unit is turned off it must remain uncommitted for at least τ^- consecutive time instants. It is therefore useful to introduce binary variables x_t indicating (if 1) the commitment of the unit at time instant t. We then define X as the set of schedules respecting minimum up- and downtime constraints; also, for any $x \in X$, we define $c(x)$ as the cost of the schedule: This may comprise fixed generating cost and time-dependent or time-invariant start-up cost. Note that because costs are usually modified by Lagrangian multipliers, the cost functions $c(x)$ and $f^t(p_t)$ may also take negative values. Other combinatorial constraints and costs could also be included, as discussed in §6, as long as they are consistent with the dynamic programming procedure discussed in the next section.

The last set of technical requirements are the ramping constraints. These require that the maximum increase of generated power from time instant t to the next be limited to $\Delta^t_+ > 0$, and, analogously, the maximum decrease of generated power from time instant t to the next be limited to $\Delta^t_- > 0$. Note that this definition can be applied only if the unit is committed both in time periods t and $t+1$. We therefore consider a general form of ramping constraints where both an upper bound $\bar{l}^t$, $l^t \leqslant \bar{l}^t \leqslant u^t$, on the maximum amount of power that can be generated if the unit is turned on in time period t (that is, it was uncommitted in $t-1$) and, analogously, an upper bound $\bar{u}^t$, $l^t \leqslant \bar{u}^t \leqslant u^t$, on the maximum amount of power that can be generated if the unit is turned off at the end of time period t (that is, it will be uncommitted in $t+1$) are known.

A formulation of (1UC) is

$$\min\ c(x) + \sum_{t \in T} f^t(p_t) \tag{1}$$

s.t.

$$l^t x_t \leqslant p_t \leqslant u^t x_t, \quad t \in T, \tag{2}$$

$$p_{t+1} \leqslant p_t + x_t \Delta^t_+ + (1 - x_t)\bar{l}^{t+1}, \quad t = 0, \ldots, n-1, \tag{3}$$

$$p_t \leqslant p_{t+1} + x_{t+1} \Delta^t_- + (1 - x_{t+1})\bar{u}^t, \quad t = 0, \ldots, n-1, \tag{4}$$

$$x \in X. \tag{5}$$

Constraints (3) are *ramp-up* constraints, i.e., they limit the maximum increase in power attainable at time instant t (assuming that the unit is committed in t). Note that we assume that we know the state of the unit at the time instant prior to the beginning of the operation, i.e., its commitment x_0 and the generated power p_0. Also, for the sake of

minimum up- and downtime constraints, we assume that we know how long the unit has been on (if $x_0 = 1$) or off (if $x_0 = 0$). Constraints (4) are *ramp-down* constraints, i.e., they limit the maximum decrease in power attainable at time instant t. We remark that this formulation is more general than those usually considered (cf. Bannister and Kaye 1991, Fan et al. 2002, Travers and Kaye 1998), not only because the cost functions need not be piecewise linear, but also because we allow different limits Δ_+^t and Δ_-^t for ramp-up and ramp-down constraints, and we allow them to depend on the time instant t.

It is well known that if constraints (3) and (4) are not present, (1UC) is easily solvable by a two-stage procedure. First, the unconstrained optimum of each f^t,

$$\tilde{p}_t = \arg\min\{f^t(p):\ p \in \mathbb{R}\} \tag{6}$$

(assumed to be unique for simplicity) is computed, and used to find the optimal power production level if the unit is committed:

$$\begin{aligned} p_t^* &= \min\{u^t, \max\{\tilde{p}_t, l^t\}\} \\ &= \arg\min\{f^t(p):\ l^t \leqslant p \leqslant u^t\}, \end{aligned} \tag{7}$$

by simply projecting $\tilde{p}_t$ over the feasible set $[l^t, u^t]$. The value $z^t = f^t(p_t^*)$ is the contribution of variable p_t to the objective function value if the unit is committed at time instant t ($x_t = 1$), while 0 is the contribution if the unit is uncommitted ($x_t = 0$). Thus, z^t is the cost (or revenue) of committing the unit, to be considered together with fixed costs and start-up costs, effectively eliminating the p_t variables from the problem. The remaining combinatorial problem in the x_t variables alone can be easily solved by dynamic programming. In a dynamic programming approach, the first step is the identification of the *state space* of all the possible solutions and partial solutions. When this space is composed of a finite number of elements, it is possible to represent it as a directed acyclic graph where the nodes are the states and the arcs identify the transitions among the states; it is then possible to find an optimal solution by computing a shortest path on this graph.

In the simple case where start-up costs are time invariant, that is, the cost of starting up the unit at a certain time instant does not depend on how long the unit has been uncommitted (but it may depend on the specific time instant), the state space of the dynamic programming is made of $2n$ nodes, say $(t, 1)$ and $(t, 0)$ for $t \in T$, representing, respectively, the unit being committed ($x_t = 1$) or not ($x_t = 0$) at time instant t, plus a source s and a sink d. There are arcs between nodes $(t, 1)$ and $(t + 1, 1)$ for all $t < n$, representing the fact that the unit, which has already passed the τ^+ periods of mandatory commitment, is kept on in time instant t, labeled with the sum of the corresponding z^t and fixed cost (if any). Analogously, there are arcs between nodes $(t, 0)$ and $(t + 1, 0)$ for all $t < n$, representing the fact that the unit, which has already passed the τ^- periods of mandatory uncommitment, is kept off in time instant t, labeled with zero cost. Then, there are arcs for state switches, i.e., arcs from $(t, 1)$ to $(t + \tau^-, 0)$, with zero cost, indicating the shutdown of the unit at time instant $t + 1$ and its remaining uncommitted for the following τ^- periods, and arcs $(t, 0)$ to $(t + \tau^+, 1)$, indicating the start-up of the unit at time instant $t + 1$ and its remaining committed for the following τ^+ periods, with the proper start-up cost plus all the generating and fixed costs for the interval. When $t + \tau^+$ (τ^-) is larger than n, the arcs go to the sink d, and the cost is properly modified. Then, there are arcs from the source s to the nodes compatible with the initial state of the unit. That is, if the unit is initially uncommitted since τ^0 time periods, there is an arc from s to $(\max\{\tau^- - \tau^0, 1\}, 0)$; if $\tau^- > \tau^0$, this indicates that the unit has to remain uncommitted for the first $\tau^- - \tau^0$ time periods. Analogously, if the unit is initially committed since τ^0 time periods, there is an arc from s to $(\max\{\tau^+ - \tau^0, 1\}, 1)$; if $\tau^0 < \tau^+$, this indicates that the unit has to remain committed for the first $\tau^+ - \tau^0$ time periods, with appropriate cost. Finally, there are zero-cost arcs from $(n, 1)$ and $(n, 0)$ to the sink d.

Clearly, every $s - d$ path on this graph represents a feasible solution to (1UC), and the cost of the path is equal to the cost of the solution. Hence, (1UC) is reduced to a shortest-path problem on an acyclic graph, which can be solved in linear time on the number of arcs, i.e., in $O(n)$.

If time-dependent start-up costs have to be considered, the graph has to be expanded somewhat, introducing nodes $(t, -k)$ indicating that the unit has remained uncommitted for the last k consecutive time instants, and properly modifying the arcs. The maximum value of k that has to be considered is the number of time instants after which the unit has completely "cooled off," i.e., a restart has the same cost as a cold start; in general, this value may be as large as n, although usually it is smaller. Thus, the size of the graph grows from $O(n)$ to $O(n^2)$ in the worst case, and the complexity of the procedure increases accordingly.

This procedure, however, fails if constraints (3) and (4) are present. In fact, the p_t variables are no longer independent once the x_t variables are fixed because they are linked together by the ramping constraints; hence, it is no longer possible to determine the optimal generated power p_t^*, and the corresponding contribution z^t of variable p_t to the objective function value, independently for each time period.

3. The Dynamic Programming Procedure

To solve (1UC) with constraints (3) and (4), a different dynamic programming procedure can be used. The state space of the dynamic programming comprises, in principle, all pairs (h, k) for $h, k \in T$ and $k \geqslant h$, plus a source s and a sink d. The meaning of each state (h, k) is: The unit is turned on at time instant h (i.e., it was uncommitted at time instant $h - 1$), and it will be turned off again at time

instant k (i.e., it will be uncommitted at time instant $k+1$). Clearly, all states such that $k < h + \tau^+ - 1$ correspond to infeasible operations and need not be constructed.

In the state-space graph G, there is an arc between node (h, k) and node (r, q) if $r \geqslant k + \tau^- + 1$, i.e., it is feasible to turn on the unit at time instant r given that it has been turned off at time instant k. Each of these arcs is labeled with the start-up cost of the unit at time instant r; note that time-dependent start-up costs are easily handled within this framework. There are also arcs from the source s to all nodes (h, k) compatible with the initial state of the unit. That is, if the unit is uncommitted since τ^0 time periods, there is an arc from s to each node (h, k) such that $h + \tau^0 - 1 \geqslant \tau^-$; these arcs are labeled with the corresponding start-up cost. If instead the unit is committed since τ^0 time periods, there is an arc from s to each node $(1, k)$ such that $k + \tau^0 \geqslant \tau^+$, labeled with zero cost. Finally, there is a zero-cost arc from each node to the sink d.

Clearly, every $s - d$ path on this graph represents a feasible solution to (1UC). By now, the cost of the path only represents the contribution of start-up costs to the objective function. Obviously, fixed generating costs (if any) can also be easily included: We can associate with each node (h, k) the sum of all fixed costs of all periods from h to k (extremes included) as cost of the node because the unit will be committed in that interval.

Furthermore, for each node (h, k), the optimal contribution of the variable generating costs, which depend on the p_t variables, can be computed in polynomial time by solving the following *economic dispatch with ramping constraints* problem for the interval $[h, k]$:

$$\min \sum_{t=h}^{k} f^t(p_t) \tag{8}$$

s.t.

$$l^t \leqslant p_t \leqslant u^t, \quad h \leqslant t \leqslant k, \tag{9}$$

$$p_h \leqslant \bar{l}^h, \tag{10}$$

$$p_{t+1} \leqslant p_t + \Delta_+^t, \quad t = h, \ldots, k-1, \tag{11}$$

$$p_t \leqslant p_{t+1} + \Delta_-^t, \quad t = h, \ldots, k-1, \tag{12}$$

$$p_k \leqslant \bar{u}^k. \tag{13}$$

We will denote problem (8)–(13) as (ED_{hk}). As all the relevant binary variables are fixed, this is an optimization problem with convex objective function and linear constraints. Hence, its optimal objective function value $z^*_{hk} = z(ED_{hk})$ can be computed in polynomial time. By summing z^*_{hk} to the weight of each node (h, k), the cost of each $s - d$ path on the graph is that of the feasible solution it represents. Hence, once again (1UC) is reduced to a shortest-path problem on an acyclic graph with $O(n^2)$ nodes and $O(n^4)$ arcs. Thus, the problem can be solved in $O(n^4)$ once all the data has been computed.

However, the complexity of the visit can be reduced by exploiting some structural properties of the state-space graph G. Consider the set of nodes (h, k) in G partitioned into levels $V_k = \{(h, k): 1 \leqslant h \leqslant k\}$ for $k \geqslant 1$ (level V_0 contains only the starting node s). From the definition of G, it immediately follows that:

- all nodes in V_k have the same set of adjacent nodes; and
- the cost of the arc between (h, k) and (r, q) depends only on k and r.

Therefore, it is possible to visit G in ascending order of level k, avoiding having to explicitly explore the forward star of all but one node for each level.

In more detail, the procedure works as follows. For each $k = 1, \ldots, n$, we keep a list $S_k \subseteq V_k$ of the reached nodes $(h, k) \in V_k$ with the label d_{hk} corresponding to the length of the shortest path found so far. S_0 contains s with label 0. For $k = 0, 1, \ldots, n$, we repeat the following steps:

- evaluate $z^*_{hk} = (ED_{hk})$ for all nodes in S_k (for $k = 0$ the result is zero, if $S_k = \varnothing$ skip to next value of k);
- find the node (h, k) in S_k with smallest value of $z^*_{hk} + d_{hk}$; and
- visit all the adjacent nodes (r, q) of (h, k) computing the new value d_{rq} as the sum of $z^*_{hk} + d_{hk}$ and the cost of the arc between (h, k) and (r, q); if the node (r, q) is visited for the first time it is inserted in S_q, otherwise its label is updated if the new value is smaller than its old value.

Clearly, the chosen order is a valid one, and the visit terminates, having determined a shortest $s - d$ path. In principle, all the $O(n^2)$ nodes of G are visited, and therefore the computation of all the corresponding z^*_{hk} values is required. However, for each k, we consider only the node (h, k) associated with the shortest path from s, so that we need only check its $O((n-k)^2)$ outgoing arcs. Therefore, the complexity of the visit is reduced to $O(n^3)$ plus the cost of solving the $O(n^2)$ convex problems (ED_{hk}), with up to n variables.

Despite the relatively small size of the problem, this may turn out to be a heavy task, especially considering that several (1UC) problems are typically solved at each one of the many iterations of Lagrangian approaches to more complex (UC) problems (cf., e.g., Bacaud et al. 2001; Belloni et al. 2003; Borghetti et al. 2001; Borghetti et al. 2003a, b; Madrigal and Quintana 1999; Zhuang and Galiana 1988). Hence, solving (ED_{hk}) efficiently—or, more to the point, solving *all* the $O(n^2)$ of them efficiently—is crucial. In the next section, we develop an efficient dynamic programming algorithm for the solution of (sequences of) (ED_{hk}).

The approach is inspired by that of Fan et al. (2002), where a similar state-space graph is defined. However, that paper was limited to (1UC) problems with piecewise-linear cost functions, and strongly used the (piecewise) linearity of the objective function to solve the economic dispatch problems; the idea was to trace, using linear duality, how the breakpoints and the slopes of the cost function change when moving from (ED_{hk}) to $(ED_{h(k+1)})$. Moreover, in that approach, ramp-up and ramp-down constraints were not allowed to depend on the time instant, and initial and final upper bounds $\bar{l}^t$ and $\bar{u}^t$ are not handled.

4. Solving the Economic Dispatch Problem

We will devise an algorithm for efficiently solving sequences of (ED_{hk}) problems for $k = h, \ldots, n$. To solve $(ED_{h(k+1)})$ by exploiting the solution of (ED_{hk}), it is necessary to introduce the parametric problem $(ED_{hk}(\bar{p}))$, i.e., the restriction to (ED_{hk}) corresponding to fixing the last variable p_k to the fixed value $\bar{p}$ (equivalently, imposing the extra constraint $p_k = \bar{p}$). We then study the properties of the optimal objective function value of $(ED_{hk}(\bar{p}))$ as a function of the parameter $\bar{p}$. To simplify matters somewhat, however, it is convenient to give a slightly different definition of the function under examination:

$$z_{hk}(\bar{p}) = \begin{cases} \min\{f^h(p_h)\text{: (9), (10), } p_h = \bar{p}\} & \text{if } h = k, \\ \min\{\sum_{t=h}^{k} f^t(p_t)\text{: (9), (10), (11), (12), } p_k = \bar{p}\} & \\ \qquad\qquad \text{otherwise.} & \end{cases}$$

That is, we allow $\bar{p}$ to assume any value in the interval $[l^k, u^k]$, even those values such that fixing $p_k = \bar{p}$ in formulation (8)–(13) would result in an infeasible problem due to the stricter upper bound imposed by constraint (13). This is done because we will use z_{hk} to compute $z_{h(k+1)}$; in the latter problem, constraint (13) corresponds to variable p_{k+1}, and therefore it is no longer binding for p_k.

We first state some general properties of the function:

Proposition 1. *The function z_{hk} is convex. Moreover, it has a piecewise nature, that is, it is finite valued only in $v+1$ intervals $[m_0, m_1], [m_1, m_2], \ldots, [m_v, m_{v+1}]$, with $l^k \leqslant m_0$, $m_{v+1} \leqslant u^k$, and $v \leqslant 2(k-h)$, in which*

$$z_{hk}(\bar{p}) = z^i(\bar{p}) \quad \text{if } \bar{p} \in [m_i, m_{i+1}] \text{ for } i = 0, 1, \ldots, v,$$

where each function z^i is the sum of at most $k-h+1$ functions f^t for $t \in \{h, h+1, \ldots, k\}$ (and therefore it is convex).

Proof. Convexity of z_{hk} is a consequence of well-known general properties that need not be discussed here beyond noting that z_{hk} is the *value function* (Hiriart-Urruty and Lemaréchal 1993) of the convex program $(ED_{hk}(\bar{p}_k))$ with respect to the right-hand side $\bar{p}_k$ of its constraint $p_k = \bar{p}_k$. Its piecewise nature, and the more specific properties, will be demonstrated next by outlining the steps for efficiently constructing the piecewise representation of z_{hk}.

We will proceed by induction to prove that the claimed properties are true; equivalently, we will (efficiently) construct piecewise representations of the functions $z_{hh}, z_{h(h+1)}, \ldots, z_{hk}$, in this order. At each step, we will exploit the previously computed representation to construct that of the next problem. During the process, for each step k, we will also (efficiently) compute and exploit

$$p^*_{hk} = \arg\min\{z_{hk}(p)\text{: } p \in [l^k, u^k]\},$$

that is, the kth (last) component of the optimal solution of (ED_{hk}) where constraint (13) is relaxed.

At the basis of the induction process, the case $k = h$ is straightforward because

$$z_{hh}(\bar{p}) = f^h(\bar{p})$$

for all $\bar{p} \in [l^h, \bar{l}^h]$. Hence, there are $v+1 = 1$ intervals and $v+1 = 1$ functions with the required properties (i.e., $0 = v \leqslant 2(h-h) = 0$). In this case, p^*_{hk} is just p^*_h as computed with formulae (6) and (7), but with the upper bound equal to $\bar{l}^h$. When $h = 1$ and the unit was already committed, the set in which $z_{11}(\bar{p})$ is defined is slightly different: Constraint (10) is not present, while p_0 and the ramping constraints must be considered, therefore $\bar{p} \in [\max\{l^1, p_0 - \Delta^1_-\}, \min\{u^1, p_0 + \Delta^1_+\}]$.

Now we assume the claim proved for some value of k—and the corresponding set of intervals and functions to have already been explicitly computed—and proceed in proving that Proposition 1 holds for $k+1$, too. We will denote $\bar{m}_i$ the extremes of the intervals for $z_{h(k+1)}$, $\bar{z}^i$ the corresponding functions, and $\bar{v}+1$ their number.

Consider any fixed value $\bar{p} \in [l^{k+1}, u^{k+1}]$. Constraints (11) and (12), written for $p_{k+1} = \bar{p}$ and p_k, result in

$$\bar{p} - \Delta^k_+ \leqslant p_k \leqslant \bar{p} + \Delta^k_-.$$

As z_{hk} is infinite valued for p_k outside $[m_0, m_{v+1}]$, one has to set $\bar{m}_0 = \max\{l^{k+1}, m_0 - \Delta^k_-\}$ and $\bar{m}_{\bar{v}+1} = \min\{u^{k+1}, m_{v+1} + \Delta^k_+\}$; in fact, $z_{h(k+1)}$ is clearly infinite valued outside this interval, and finite valued inside it. Note that if $\bar{m}_0 > \bar{m}_{\bar{v}+1}$, infeasibility of the (1UC) problem has been detected.

Now consider how the optimal solution to $(ED_{h(k+1)}(\bar{p}))$ can be computed, exploiting the (already computed) piecewise representation of z_{hk}. The problem can clearly be rewritten as

$$z_{h(k+1)}(\bar{p}) = f^{k+1}(\bar{p}) + \min\{z_{hk}(p_k)\text{: } m_0 \leqslant p_k \leqslant m_{v+1}, \ \bar{p} - \Delta^k_+ \leqslant p_k \leqslant \bar{p} + \Delta^k_-\}.$$

In other words, the optimal solution to $(ED_{h(k+1)}(\bar{p}))$—at least, its kth component—is just the constrained minimum of z_{hk} in the intersection of the intervals $[\bar{p} - \Delta^k_+, \bar{p} + \Delta^k_-]$ and $[m_0, m_{v+1}]$; we will denote that minimum as $p^*_k(\bar{p})$. To trace the function $z_{h(k+1)}(\bar{p})$, it is only necessary to understand how $p^*_k(\bar{p})$ behaves as $\bar{p}$ changes. This is, however, very simple to picture.

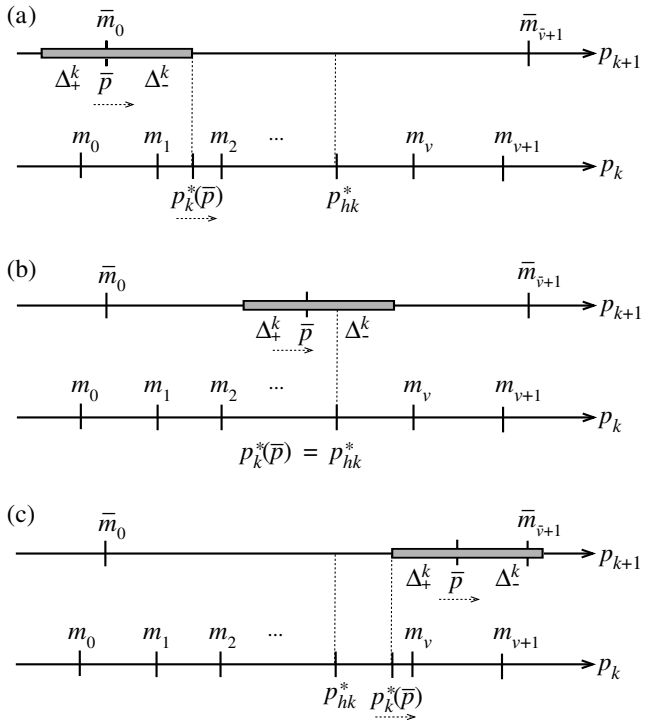
Consider the constrained minimum p^*_{hk} of z_{hk} over $[m_0, m_{v+1}]$ (or, equivalently, $[l^k, u^k]$), that we assume to have already computed: Because z_{hk} is convex, $p^*_k(\bar{p})$ is just its projection over the feasible interval

$$[\bar{p} - \Delta^k_+, \bar{p} + \Delta^k_-] \tag{14}$$

(cf. (7)), that is,

$$p^*_k(\bar{p}) = \min\{\bar{p} + \Delta^k_-, \max\{p^*_{hk}, \bar{p} - \Delta^k_+\}\}. \tag{15}$$

Figure 1. Evolution of $p_k^*(\bar{p})$ as $\bar{p}$ varies.



Note that we are assuming p_{hk}^* to be unique; however, the following arguments can be easily extended to cases where z_{hk} has a (known) nonpointed interval as the set of optimal solutions.

We can pictorially describe the process as follows, with the help of Figure 1. Basically, as $\bar{p}$ varies from $\overline{m}_0$ to $\overline{m}_{\bar{v}+1}$, three different cases can arise:

(a) When $\bar{p}$ is on the leftmost part of the interval $[\overline{m}_0, \overline{m}_{\bar{v}+1}]$ where $z_{h(k+1)}$ is finite valued, e.g., $\bar{p} = \overline{m}_0$, p_{hk}^* is "on the right" of the feasible interval (14) (i.e., $p_{hk}^* > \bar{p} + \Delta_-^k$), $p_k^*(\bar{p}) = \bar{p} + \Delta_-^k$, that is, $p_k^*(\bar{p})$ is a linear function of $\bar{p}$.

(b) As $\bar{p}$ increases, eventually p_{hk}^* falls inside the feasible interval (14): Then, $p_k^*(\bar{p})$ is equal to p_{hk}^*, remaining fixed until $\bar{p}$ becomes too large.

(c) Finally, $\bar{p}$ becomes larger than $p_{hk}^* + \Delta_+^k$, that is, p_{hk}^* no longer falls inside of the feasible interval (14), this time remaining "on the left;" then $p_k^*(\bar{p}) = \bar{p} - \Delta_+^k$, so, again, $p_k^*(\bar{p})$ increases linearly as $\bar{p}$ does.

Of course, all three cases (a), (b), and (c) need not necessarily happen. For instance, p_{hk}^* may already belong to (14) for $\bar{p} = \overline{m}_0$, or it may never leave it even if $\bar{p} = \overline{m}_{\bar{v}+1}$, and so on. However, the above three cases cover all that can possibly happen.

It is now easy to see how, given the explicit description of z_{hk} in terms of the $v+1$ subintervals of $[m_0, m_{v+1}]$ and the associated functions z^i, we can efficiently construct a piecewise representation of $z_{h(k+1)}$ with $\bar{v}+1$ intervals where $\bar{v} \leqslant v+2$.

Step 0. Set $\bar{p} = \overline{m}_0$, $\bar{v} = 0$, and let $0 \leqslant q \leqslant v$ be the index of the interval to which $p_k^*(\bar{p}) = \bar{p} + \Delta_-^k$ belongs (if it is a break point, choose the interval on the right). Set $\bar{u} = \min\{u^{k+1}, m_{v+1} + \Delta_+^k\}$.

Step 1. If case (a) is not verified, go to Step 2, otherwise, set $\bar{z}^{\bar{v}}(p) = f^{k+1}(p) + z^q(p + \Delta_-^k)$. Compute the maximum value of $\bar{p}$ such that $p_k^*(\bar{p})$ remains in the qth interval, p_{hk}^* remains outside the feasible interval, and $\bar{p}$ remains feasible, that is, $\bar{p} = \min\{m_{q+1} - \Delta_-^k, p_{hk}^* - \Delta_-^k, \bar{u}\}$. Set $\bar{v} = \bar{v} + 1$, $\overline{m}_{\bar{v}} = \bar{p}$, if $\bar{p} \neq p_{hk}^* - \Delta_-^k$, then $q = q + 1$, and repeat Step 1.

Step 2. If case (b) is not verified, go to Step 3, otherwise set $\bar{z}^{\bar{v}}(p) = f^{k+1}(p) + z^q(p_{hk}^*)$. Compute the maximum value of $\bar{p}$ such that p_{hk}^* remains inside the feasible interval and $\bar{p}$ remains feasible, that is, $\bar{p} = \min\{p_{hk}^* + \Delta_+^k, \bar{u}\}$. Set $\bar{v} = \bar{v} + 1$, $\overline{m}_{\bar{v}} = \bar{p}$, and go to Step 3.

Step 3. If $\bar{p} = \bar{u}$, then terminate, otherwise set $\bar{z}^{\bar{v}}(p) = f^{k+1}(p) + z^q(p - \Delta_+^k)$. Compute the maximum value of $\bar{p}$ such that $p_k^*(\bar{p}) = \bar{p} - \Delta_+^k$ remains in the qth interval and $\bar{p}$ remains feasible, that is, $\bar{p} = \min\{m_{q+1} + \Delta_+^k, \bar{u}\}$. Set $\bar{v} = \bar{v} + 1$, $\overline{m}_{\bar{v}} = \bar{p}$, $q = q + 1$, and repeat Step 3.

Clearly, the total number of intervals for $z_{h(k+1)}$ is at most equal to that for z_{hk} plus the two ones corresponding to p_{hk}^* "entering" and "leaving" the feasible set, that is, the former interval q in Step 2 is replaced by at most three new intervals. Note that the intervals with right extreme less than or equal to $p_{hk}^* - \Delta_-^k$, if any, correspond to intervals for z_{hk} "shifted left" by Δ_-^k, while the intervals with left extreme greater than or equal to $p_{hk}^* + \Delta_+^k$, if any, correspond to intervals for z_{hk} "shifted right" by Δ_+^k. The final number of intervals may well be strictly less than $v+3$. Also, as each z^i is composed of the sum of at most $k-h+1$ original functions f^t, each $\bar{z}^j$ is composed of the sum of at most $k-h+2 = (k+1)-h+1$ original functions f^t. This completes the proof of Proposition 1. □

During the above process, it is very easy to compute not only $p_{h(k+1)}^*$, but also the optimal solution of

$$\min\{z_{h(k+1)}(p):\ p \in [l^{k+1}, \bar{u}^{k+1}]\},$$

that is, the last component of the optimal solution of $(ED_{h(k+1)})$, where constraint (13) is imposed. Thus, the above procedure can be used to solve (ED_{hk}) problems.

The complexity of the procedure depends on the actual form of the functions f^t; if the functions are quadratic, as is common in practical applications, each step of the procedure is $O(1)$. Therefore, assuming that $(ED_{h(k-1)})$ has already been solved (with the same method), the complexity to solve (ED_{hk}) is $O(k-h)$, and, consequently, the complexity to solve all the problems (ED_{hh}), $(ED_{h(h+1)}), \ldots, (ED_{hk})$ is $O((k-h)^2)$. Hence, solving all the $O(n^2)$ (ED) problems in the dynamic programming procedure of the previous paragraph has $O(n^3)$ complexity. All in all, combining the special visit of the state-space graph G with the above efficient procedure for solving (ED_{hk}), we can solve (1UC), for the quadratic case, in $O(n^3)$. The same complexity bound holds for any other class of convex functions closed under the sum operation and such

that an $O(1)$ closed-form formula exists for computing the unconstrained minima, such as, among others, polynomial functions of degree at most five. The approach is, however, likely to prove efficient even for other classes of functions because univariate unconstrained optimization approaches can be used to compute the required unconstrained minima. This is likely to be much more efficient than the corresponding multivariate constrained optimization approaches required to solve (ED_{hk}) as a whole.

Although the optimal objective function value of each (ED_{hk}) problem is all that is needed for solving (1UC), the optimal solutions are then required to reconstruct a full optimal solution—both in the commitment variables and in the power variables—of the problem. More specifically, the optimal solutions of all the (ED_{hk}) problems corresponding to all nodes in the optimal path are needed. However, those solutions are easily found with a "backward pass," using the available information constructed in the "forward pass." In fact, consider a given problem (ED_{hk}). As discussed above, the optimal value of the last variable p_k, say $\bar{p}_k$, is available when the problem is solved. Then, it is immediate to compute the optimal value $\bar{p}_{k-1}$ of the previous variable p_{k-1} (if $k > h$) by just computing the projection of the available constrained minimum $p^*_{h(k-1)}$ (of $z_{h(k-1)}$, over $[l^{k-1}, u^{k-1}]$) onto $[\bar{p}_k - \Delta_+^{k-1}, \bar{p}_k + \Delta_-^{k-1}]$. Iterating this procedure, the whole solution of (ED_{hk}) can clearly be found in $O(k-h)$. As only the optimal solution of the relevant (ED_{hk}) problems—those corresponding to nodes in the optimal path—is required, and the total number of time instants in which the unit is committed in those nodes is at most n, the optimal solution to (1UC), in terms of the power variables, can be found in $O(n)$.

It is easy to see that the dual optimal solution to each (ED_{hk}) can also be constructed, during the "backward pass," together with the optimal primal solution. In fact, each $\bar{p}_t$ for $h \leqslant t \leqslant k$ is the constrained minimum of z_{ht} subject to (9)–(13), which ultimately defines a nonempty interval in the real line. Thus, if $\bar{p}_t$ lies in the strict interior of the interval, i.e., none of the constraints is active, then all the corresponding optimal dual variables are zero. Assume instead that exactly one constraint, say (11), is active in $\bar{p}_t$ and that z_{ht} is differentiable in $\bar{p}_t$ (the argument can be easily extended to the case of multiple active constraints). The Karush-Kuhn-Tucker conditions require that

$$z'_{ht}(\bar{p}_t) = \pi,$$

where π is the optimal dual variable of (11). Similar formulae can be easily derived for all other constraints. Hence, optimal dual information is readily available at the only cost of computing the derivative of z_{ht}. In the quadratic case, where this is $O(1)$, the total cost of retrieving the dual optimal solution to (ED_{hk}) is $O(k-h)$. Clearly, the above technique can be extended to a nondifferentiable z_{ht}; only left and right derivatives are to be computed.

Somewhat surprisingly, it does not appear that the procedure can be significantly streamlined or simplified if further assumptions are made on the data. For instance, in many practical applications one has $l^t = l$, $u^t = u$, $\Delta^t_- = \Delta^t_+ = \Delta < u - l$ (for if $\Delta \geqslant u - l$, then ramping constraints are redundant), $\bar{l}^t = \bar{u}^t = l + \Delta$ for all $t \in T$, and $f^t(p) = ap^2 + b_t p$ (that is, only the linear part of the quadratic objective function depends on the time instant). However, it does not appear that the worst-case complexity results can be improved even if all the above assumptions are made.

However, it is possible to improve the performances of the method in practice by avoiding (building and) visiting all the state-space graph G of the dynamic programming procedure. This can be done by observing that every arc and node in G represents a certain number of (consecutive) time instants in T, and each $s-d$ path in G ultimately represents exactly n time instants. Thus, adding to the cost of each arc and node a quantity proportional to the number of time instants it covers, say M times the number of time instants, where M is the same for all arcs and nodes, the cost of every $s-d$ path increases by Mn, and therefore the optimal solution does not change. Actually, one may even define a different value M_t for each $t \in T$ and add it to each node/arc that contains t. This allows us to make the cost of every arc and node in G nonnegative by simply choosing M large enough. In typical applications, we do not even need to compute the actual cost of every arc and node for being able to compute a suitable value for M; in fact, only the costs of the nodes can be negative, hence, computing $z^*_t = \min\{f^t(p)\colon\ p \in [l^t, u^t]\}$ (cf. (7)) and setting $M = -\min\{\min\{z^*_t\colon\ t \in T\}, 0\}$, one ensures that all the resulting node (and arc) costs are nonnegative.

Then, knowing the objective function value of one—hopefully, good—solution, that is, the cost of one $s-d$ path, it may be possible to terminate early the visit of some part of the graph, avoiding generating some of its nodes and the corresponding arcs. In fact, having all arc and node costs made nonnegative, the cost of any partial $s-d$ path cannot be smaller than the cost of any $s-d$ path containing it. Thus, if a partial path is found whose cost is larger than that of the best-known solution, the visit of the graph from its last node can be interrupted. In a Lagrangian setting, a reasonable choice for the initial incumbent $s-d$ path could be the optimal solution of the (1UC) problem corresponding to the same unit at the previous Lagrangian iteration. Of course, as soon as a better $s-d$ path is found during the visit, the value of the incumbent can be updated.

5. Computational Results

To test the actual efficiency of the proposed approach, we implemented it and compared it with the CPLEX 8.0 general-purpose mixed-integer quadratic programming solver. We remark that our code could surely be improved, for instance, by the preprocessing techniques described at the end of the previous paragraph; however, even this somewhat preliminary implementation has already obtained satisfactory results.

To test the approach, we considered three (UC) problems, each with the same 100 thermal units but with different values of n: 24, 96, and 168. We solved the problems with the Lagrangian approach of Borghetti et al. (2003a) that in all three cases performed 23 iterations to reach the optimal solution of the Lagrangian dual with respect to demand constraints and spinning-reserve constraints. We recorded the dual prices of these constraints at four particular iterations, which can be considered a representative sample: 1, 12, 16, and 23, that is, the first one, the last one, and two intermediate iterations. This gave us a set of $100 \times 3 \times 4 = 1{,}200$ (1UC) problems, each one corresponding to a specific unit, a given time horizon, and one of the iterations of the Lagrangian approach; we solved all these instances both with our implementation of the proposed approach and with CPLEX 8.0 on a Pentium IV 2.5 GHz processor with 1.5 GB of RAM under Debian Linux 3.0. Our code was compiled with g++ version 3.0.4 and optimization option -O3; CPLEX was given a maximum time limit of 300 seconds to solve each instance.

The results of this computational experience are summarized in Table 1. Each row is associated with the 100 instances of a particular Lagrangian iteration and time horizon. In the first two columns, we report the average solution time and the relative standard deviation obtained with the dynamic programming approach. In the following two columns, we report the same information for the CPLEX 8.0 MIQP solver; because the latter was not always able to solve the instance within its time limit, in the last two columns we also report the average gap at termination (zero when not shown) and the number of instances that could not be solved by CPLEX at optimality.

The dynamic programming algorithm solves all instances, on average, much faster than CPLEX (almost three orders of magnitude faster on the largest instances), with a negligible standard deviation. For the largest instances, CPLEX was not able to solve a significant fraction of the instances to optimality within the time limit; the average gap of 1% on these instances means that for some unsolved instance the gap was greater than 8%. Also, CPLEX shows a very high standard deviation, meaning that while some instances were solved relatively fast (mainly due to a good preprocessing phase), others took a very long time. All in all, these results show that while implementing a Lagrangian approach to ramp-constrained (UC) by solving the (1UC) subproblems with CPLEX is hardly feasible, the proposed dynamic programming algorithm can solve the subproblems efficiently enough.

Table 1. Computational results of the DP algorithm vs. the CPLEX MIQP solver.

Instance		DP		CPLEX			
n	Iter.	Time	Std. dev.	Time	Std. dev.	Gap%	Fail
24	1	0.001	3e−3	0.05	0.05		0
	12	0.002	4e−3	0.08	0.05		0
	16	0.002	4e−3	0.08	0.05		0
	23	0.002	4e−3	0.08	0.05		0
96	1	0.04	2e−3	10.74	41.99		1
	12	0.04	3e−3	17.57	50.93	0.06	2
	16	0.04	2e−3	32.64	76.87	0.02	6
	23	0.04	3e−3	32.21	76.12	0.03	6
168	1	0.20	6e−3	47.73	103.68	1.09	13
	12	0.20	6e−3	117.94	142.61	1.20	35
	16	0.20	5e−3	117.49	142.11	0.50	35
	23	0.20	6e−3	117.46	141.87	1.23	35

6. Conclusions

We have proposed an efficient dynamic programming algorithm for solving (1UC) with ramping constraints and general convex cost functions. The algorithm requires solving $O(n^2)$ convex programs, with up to n variables each, to compute the data for the dynamic programming procedure; the main contribution of this paper is precisely the proposal of a new efficient algorithm for solving these problems. The resulting algorithm is simple to implement and works for a very general form of (1UC) with time-varying upper and lower limits over the generated power, as well as time-varying and different limits for ramp-up and ramp-down constraints. Coupled with a special visit of the state-space graph in the dynamic programming algorithm, this enables one to solve (1UC) in $O(n^3)$ overall for suitable cost functions, such as quadratic ones. Computational results showed that even a preliminary implementation of the proposed approach is indeed efficient, and it clearly outperforms the MIQP solver of CPLEX 8.0 for solving (1UC).

It is worth noting that the proposed approach can be extended to more general versions of (1UC) as well:

• *Data dependent on the history of the unit*. It is easy to see that the approach immediately extends, with almost no change, to problems where the data of (ED_{hk})—(coefficient of the) cost functions, coefficients of the ramping constraints, maximum and minimum production levels—depend not only on t, but on h as well, that is, on how long the unit has been committed. This may be useful, e.g., to exploit better data fitting for the coefficients of the cost functions to more accurately reflect the true operational cost of the unit. Note that a "monolithic" integer nonlinear programming model implementing this feature would be significantly larger than (1)–(5), and therefore significantly more difficult to solve by standard means, while our approach handles this generalization without increasing the computational cost.

• *Different discretization intervals for commitment and power variables*. In some cases, one may want to use different—typically, finer—discretization intervals for power variables than for commitment decisions. This may be due either to specific regulations of the operating context or to the need to better reflect the operating characteristics of the unit. It is easy to see that our approach can be easily extended to handle this case as well; the total complexity becomes $O(m^2 n)$, where m $(\geqslant n)$ is the number of power variables.

Acknowledgment

The authors are grateful to Fabrizio Lacalandra for useful discussions on these topics.

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