Note Title 2021-04-27

$A \in \mathbb{R}^{n\times n}$ invertibile $\quad A = M - N$, M invertibile

$$x_{k+1} = \underbrace{M^{-1}N}_{P} x_k + \underbrace{M^{-1}b}_{q} \qquad x_0 \in \mathbb{R}^n \text{ dato}$$

Jacobi: $M = \mathrm{diag}(A)$

esempio $n=4$

$$M x_{k+1} = N x_k + b$$

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & 0 & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & 0 & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \\ x_4^{(k)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{cases} a_{11} x_1^{(k+1)} = \left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right) / a_{11} \\ a_{22} x_2^{(k+1)} = \left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)}\right) / a_{22} \\ a_{33} x_3^{(k+1)} = \left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - a_{34}x_4^{(k)}\right) / a_{33} \\ a_{44} x_4^{(k+1)} = \left(b_4 - a_{41}x_1^{(k)} - a_{42}x_2^{(k)} - a_{43}x_3^{(k)}\right) / a_{44} \end{cases} \quad \text{for } i = 1:n$$

$j = 1:n$, saltando $j = i$

$j = 1:i-1 \qquad j = i+1:n$

$x_0 \to x_1$ (xold, xnew)

$x_1 \to x_2$ (xold, xnew)

Criterio di arresto: $\|x_k - x_*\|$ (sol. esatta, non nota a priori)

1) $\|Ax_k - b\|$ piccolo

2) $\|x_{k+1} - x_k\|$ piccolo

$$x_1^{old} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{old} - a_{13}x_3^{old}\right)$$

($a_{11} = 0$; $-\inf$; $2.2632 \cdot 10^{307}$)

$P = M^{-1}N$

Condizione sufficiente per la convergenza è $\|P\| < 1$

Condiz. necessaria e sufficiente è $\rho(P) < 1$

Gauss-Seidel: $M = \text{tril}(A)$

$M x^{k+1} = N x^k + b \qquad n=4$

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ 0 & 0 & -a_{23} & -a_{24} \\ 0 & 0 & 0 & -a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \\ x_4^{(k)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$a_{11} x_1^{(k+1)} = b_1 - a_{12} x_2^{(k)} - a_{13} x_3^{(k)} - a_{14} x_4^{(k)}$$

$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = b_2 - a_{23} x_3^{(k)} - a_{24} x_4^{(k)}$$

$$a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k+1)} = b_3 - a_{34} x_4^{(k)}$$

$$a_{41} x_1^{(k+1)} + a_{42} x_2^{(k+1)} + a_{43} x_3^{(k+1)} + a_{44} x_4^{(k+1)} = b_4$$

$$x_i^{(k+1)} = \frac{b_i - a_{i,i+1} x_{i+1}^{(k)} - a_{i,i+2} x_{i+2}^{(k)} - \dots - a_{i,n} x_n^{(k)} - a_{i1} x_1^{(k+1)} - a_{i2} x_2^{(k+1)} - \dots - a_{i,i-1} x_{i-1}^{(k+1)}}{a_{ii}}$$

Gauss-Seidel "con una variabile sola"

for $i = 1:n$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

end

$$x = \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{bmatrix} \xrightarrow{\text{passo } i=1} \begin{bmatrix} x_1^{k+1} \\ x_2^k \\ x_3^k \\ x_4^k \end{bmatrix} \xrightarrow{\text{passo } i=2} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+2)} \\ x_3^{(k)} \\ x_4^{(k)} \end{bmatrix} \xrightarrow{\text{passo } i=3} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k)} \end{bmatrix} \xrightarrow{\text{passo } i=4} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k+1)} \end{bmatrix}$$

$$e_k = P^k e_0 \qquad \|e_k\| \leq \|P\|^k \|e_0\|$$