Note Title 2021-04-13

(n=4)

$$\begin{bmatrix} 1 & -2 & & \\ & 1 & -2 & \\ & & 1 & -2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \to \begin{cases} x_1 - 2x_2 = b_1 \\ x_2 - 2x_3 = b_2 \\ x_3 - 2x_4 = b_3 \\ x_4 = b_4 \end{cases} \quad \begin{cases} x_1 = b_1 + 2x_2 \\ x_2 = b_2 + 2x_3 \\ x_3 = b_3 + 2x_4 \\ x_4 = b_4 \end{cases}$$

In sim. generica, analogamente,

$$x_n = b_n$$
$$\left.\begin{array}{l} x_{n-1} = b_{n-1} + 2x_n \\ \vdots \\ x_1 = b_1 + 2x_2 \end{array}\right\} \text{in un ciclo for}$$

Complessità:

```
x(n)=b(n)                      ← 0 op
for k = n-1:-1:1
      x(k) = b(k) + 2 * x(k+1)    ← 2 op
end
```

$2(n-1) = 2n-2$ operazioni

$O(n)$ $\quad 2n + O(1)$

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x = sup_solve_T(b)

Tn = genera_matrice_Tn(n);

x = sup_solve(T, b); $\quad$ → $O(n^2)$ ⚠ no!

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = b_1 \\ x_2 + x_3 + x_4 + x_5 = b_2 \\ x_3 + x_4 + x_5 = b_3 \\ x_4 + x_5 = b_4 \\ x_5 = b_5 \end{cases}$$

$$\begin{cases} x_5 = b_5 \\ x_4 = b_4 - (x_5) \\ x_3 = b_3 - (x_4 + x_5) \\ x_2 = b_2 - (x_3 + x_4 + x_5) \\ x_1 = b_1 - (x_2 + x_3 + x_4 + x_5) \end{cases}$$

$k = n-1$ somme $= x_5$

somme $= x_5 + x_4 + x_5$ ⚠

genericamente, $x_i = b_i - (x_{i+1} + x_{i+2} + \dots + x_n)$ $\quad i = n-1, n-2, \dots 1$

```
for k=1:n-1
    somme=0                    0 op
    for j=k+1:n
        somme=somme+x(j)       1 op     n-k volte
    end                                 n-k operaz.
    x(k)=b(k)-somme            1 op
end
```

for $k = 1:n-1$

$n-k+1$ operazioni

cioè

$n + (n-1) + (n-2) + \dots + 3 + 2$

$= \frac{n(n+1)}{2} - 1$

$= \frac{n^2}{2} + O(n)$

$= O(n^2)$

↑ Troppe!

In realtà, al passo k,

somme $= x_{k+1} + \underbrace{x_{k+2} + x_{k+3} + \dots + x_n}$

già calcolato al passo precedente

Primo passo: somma dovrà valere $x(n)$, alla fine, cioè $x(k+1)$

```
for k = n-1 : -1 : 1
    somma = somma + x(k+1)
    x(k) = b(k) - somma
end
```

} 2 op.

$2(n-1)$ operazioni

$O(n)$

$$\begin{cases} x_5 = b_5 \\ x_4 = b_4 - (x_5) \\ x_3 = b_3 - (x_4 + x_5) \\ x_2 = b_2 - (x_3 + x_4 + x_5) \\ x_1 = b_1 - (x_2 + x_3 + x_4 + x_5) \end{cases}$$

$x_5 = b_5$ (somma = 0)

in un for:

→ somma = x_5 + somma; $x_4 = b_4$ − somma

→ somma = x_4 + somma; $x_3 = b_3$ − somma

→ somma = x_3 + somma; $x_2 = b_2$ − somma

⋮

Matrice tridiagonale:

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & 0 \cdots & 0 \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & 0 \; 0 \cdots & 0 \\ 0 & 0 & \gamma_3 & \alpha_4 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \beta_{n-1} \\ 0 & \cdots & & 0 & \gamma_{n-1} & \alpha_n \end{bmatrix}$$

voglio L_1 tale che $L_1 T = \begin{bmatrix} \alpha_1 & \\ 0 & \\ 0 & \circledast \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$

$e_1^T = [1 \; 0 \; 0 \; \cdots \; 0]$

devo scegliere $L_1 = \begin{bmatrix} 1 & & & \\ -v_{21} & 1 & & \\ -v_{31} & & 1 & \\ \vdots & & & \ddots \\ -v_{n1} & & & & 1 \end{bmatrix}$ $\quad L_1 = I - v_1 e_1^T \quad v_1 = \begin{bmatrix} 0 \\ v_{21} \\ v_{31} \\ \vdots \\ v_{n1} \end{bmatrix}$

$$L_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_{k1} = \frac{x_k}{x_1} \quad \text{per ogni } k = 2, 3, \dots n$$

$$v_{21} = \frac{\gamma_1}{\alpha_1} \qquad v_1 = \begin{bmatrix} 0 \\ \gamma_1/\alpha_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 1 & & & & \\ -\gamma_1/\alpha_1 & 1 & & 0 & \\ 0 & & 1 & & \\ 0 & & & 1 & \\ \vdots & 0 & & & \ddots \\ 0 & & & & & 1 \end{bmatrix}$$

$$v_{31} = \frac{0}{\alpha_1} = 0$$

$$\vdots$$

$$v_{n1} = \frac{0}{\alpha_1} = 0$$

$$L_1 T = \begin{bmatrix} 1 & & & \\ -\gamma_1/\alpha_1 & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \cdots 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 0 & 0 & 0 \cdots 0 \\ & \gamma_2 & \alpha_3 & \beta_3 & & \\ & & \gamma_3 & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \beta_{n-1} \\ & & & & \gamma_{n-1} & \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 - \frac{\gamma_1}{\alpha_1}\beta_1 & \beta_2 & 0 & 0 \cdots & 0 \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & \beta_{n-1} \\ 0 & & & & \gamma_{n-1} & \alpha_n \end{bmatrix}$$

$$\hat{\alpha}_2 := \alpha_2 - \frac{\gamma_1}{\alpha_1}\beta_1$$

$$T_2 = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & & \\ 0 & \hat{\alpha}_2 & \beta_2 & 0 & & \\ 0 & \gamma_2 & \alpha_3 & \beta_3 & & \\ \vdots & 0 & \gamma_3 & \ddots & \ddots & \\ 0 & 0 & & \ddots & \ddots & \beta_{n-1} \\ & \vdots & & & & \\ 0 & 0 & & & \gamma_{n-1} & \alpha_n \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -v_{32} & \ddots & \\ & -v_{42} & & \ddots \\ & \vdots & & \\ & -v_{n2} & & & 1 \end{bmatrix} = I - v_2 e_2^T$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ v_{32} \\ v_{42} \\ \vdots \\ v_{n2} \end{bmatrix}$$

$$L_2 \cdot \begin{bmatrix} \beta_1 \\ \hat{\alpha}_2 \\ \gamma_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \hat{\alpha}_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ \gamma_2/\hat{\alpha}_2 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad e_2^T = [0 \ 1 \ 0 \ 0 \cdots 0]$$

(Lemma generale: dato $x \in \mathbb{R}^n$ con $x_k \neq 0$, se scelgo $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_{k+1}/x_k \\ \vdots \\ x_n/x_k \end{bmatrix}$

ho che $(I - v e_k^T) x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$)

$$T_3 = L_2 T_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ 0 & -\gamma_2/\hat{\alpha}_2 & 1 & & \\ & 0 & & \ddots & \\ & 0 & & & \\ & \vdots & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & \hat{\alpha}_2 & \beta_2 & 0 & 0 \cdots & 0 \\ & \gamma_2 & \alpha_3 & \beta_3 & 0 \cdots & 0 \\ & & \ddots & \ddots & \ddots & \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & & 0 \\ 0 & \hat{\alpha}_2 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \hat{\alpha}_3 & \beta_3 & 0 & 0 & 0 \cdots 0 \\ & & \gamma_3 & \alpha_4 & \beta_4 & & \\ & & 0 & \ddots & \ddots & \ddots & \\ & & \vdots & & & & \\ & & 0 & & & & \end{bmatrix}$$

x

$$\hat{\alpha}_3 = -\frac{\gamma_2}{\hat{\alpha}_2}\beta_2 + \alpha_3$$

$$L_3 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -\gamma_3/\hat{\alpha}_3 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \qquad L_3 T_3 = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & & & \\ 0 & \hat{\alpha}_2 & \beta_2 & & & \\ 0 & 0 & \hat{\alpha}_3 & \beta_3 & & \\ 0 & 0 & 0 & \hat{\alpha}_4 & \beta_4 & \\ \vdots & \vdots & 0 & \gamma_4 & \alpha_5 & \beta_5 \\ & & \vdots & & \ddots & \ddots \\ 0 & 0 & 0 & & & \end{bmatrix}$$

$$\hat{\alpha}_4 = -\frac{\gamma_3}{\hat{\alpha}_3}\beta_3 + \alpha_4$$

Quindi, dopo n-1 passi

$$T_n = \underbrace{\begin{bmatrix} \hat{\alpha}_1 & \beta_1 & & & \\ & \hat{\alpha}_2 & \beta_2 & & \\ & & \hat{\alpha}_3 & \beta_3 & \\ & & & \ddots & \ddots \\ & & & \ddots & \beta_{n-1} \\ & & & & \hat{\alpha}_n \end{bmatrix}}_{U} \qquad U = T_n = \underline{L_{n-1} \cdots L_4 L_3 L_2 L_1} T$$

$$L_k = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{\gamma_k}{\hat{\alpha}_k} & \ddots \\ & & & & 1 \end{bmatrix}$$

con

$$\hat{\alpha}_1 = \alpha_1$$

$$\hat{\alpha}_k = \alpha_k - \frac{\gamma_{k-1}}{\hat{\alpha}_{k-1}}\beta_{k-1} \quad \text{per } k = 2 \ldots n$$

$-\frac{\gamma_k}{\hat{\alpha}_k}$ in posizione $(k+1, k)$

$$U = T_n = L_{n-1} L_{n-2} \cdots L_2 L_1 T$$

$$\underbrace{L_1^{-1} L_2^{-1} \cdots L_{n-3}^{-1} L_{n-2}^{-1} L_{n-1}^{-1}}_{L} U = T$$

$$L = \begin{bmatrix} 1 & & & & \\ \gamma_1/\hat{\alpha}_1 & 1 & & & \\ 0 & \gamma_2/\hat{\alpha}_2 & 1 & & \\ 0 & 0 & \frac{\gamma_3}{\hat{\alpha}_3} & 1 & \\ \vdots & 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & \frac{\gamma_3}{\hat{\alpha}_3} & 1 \end{bmatrix}$$

Abbiamo mostrato che la fattorizz. LU di

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & \\ & \gamma_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \gamma_{n-1} & \alpha_n \end{bmatrix}$$

è pari a

$$T = LU = \begin{bmatrix} 1 & & & & 0 \\ \hat{\gamma}_1 & 1 & & & \\ & \hat{\gamma}_2 & 1 & & \\ & & \hat{\gamma}_3 & \ddots & \\ 0 & & & \ddots & \ddots \\ & & & \hat{\gamma}_{n-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{\alpha}_1 & \beta_1 & & & 0 \\ 0 & \hat{\alpha}_2 & \beta_2 & & \\ \vdots & 0 & \hat{\alpha}_3 & \ddots & \\ \vdots & \vdots & 0 & \ddots & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \hat{\alpha}_n \end{bmatrix}$$

con $\hat{\alpha}_1 = \alpha_1$,

$$\begin{cases} \hat{\alpha}_k = \alpha_k - \dfrac{\gamma_k \beta_{k-1}}{\hat{\alpha}_{k-1}} & \text{per } k = 2, \dots n \\ \hat{\gamma}_k = \dfrac{\gamma_k}{\hat{\alpha}_k} & \text{per } k = 1, 2, \dots n-1 \end{cases}$$

(in rosso: $\dfrac{\gamma_k}{\hat{\alpha}_{k-1}} = \hat{\gamma}_{k-1}$)

Vorremmo scrivere una function che calcola $\hat{\alpha}$, $\hat{\gamma}$ a partire da α, β, γ

```
function [alphahat, gammahat] = thomas(alpha, beta, gamma)
```