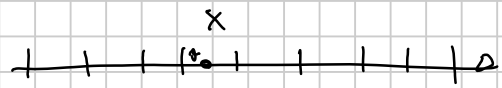
Note Title 2021-03-09

Calcolo dell'errore $\boxed{f(x) = x^2 - 1}$



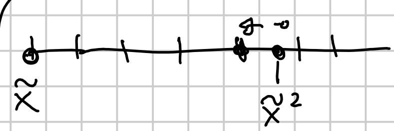
$$\tilde{x} = x(1+\epsilon_x) \qquad |\epsilon_x| \le u = 2^{-53} \approx 10^{-16}$$

$$f(\tilde{x}) = \tilde{x}^2 - 1 = x^2(1+\epsilon_x)^2 - 1$$

"errore inerente" $\dfrac{|f(\tilde{x}) - f(x)|}{|f(x)|}$

$$= x^2(1+2\epsilon_x+\underbrace{\epsilon_x^2}_{\le 10^{-32}}) - 1 \doteq x^2(1+2\epsilon_x) - 1$$

$$\frac{f(\tilde{x}) - f(x)}{f(x)} \doteq \frac{x^2(\cancel{1}+2\epsilon_x) - \cancel{1} - (\cancel{x^2} - \cancel{1})}{x^2 - 1} = \underbrace{\boxed{\frac{2x^2}{x^2-1}}}_{= \frac{f'(x)}{f(x)} \cdot x} \cdot \epsilon_x$$



$$\delta(\tilde{x}) = \underbrace{\tilde{x} \odot \tilde{x}}_{=} \ominus 1$$

$$= \overbrace{\tilde{x} \cdot \tilde{x}(1+\delta_1)} \ominus 1$$

$|\delta_1|, |\delta_2| \le u$

$$= (\tilde{x}^2(1+\delta_1) - 1)(1+\delta_2)$$

$$= \tilde{x}^2(1+\delta_1)(1+\delta_2) - 1 - \delta_2$$

$$= \tilde{x}^2(1+\delta_1+\delta_2+\underbrace{\delta_1\delta_2}_{\le 10^{-32}}) - 1 - \delta_2 \doteq \tilde{x}^2(1+\delta_1+\delta_2) - 1 - \delta_2$$

$$\text{err. alg.} = \frac{|g(\tilde{x}) - f(\tilde{x})|}{|f(\tilde{x})|} = \frac{\tilde{x}^2(\cancel{1}+\delta_1+\delta_2) - \cancel{1} - \delta_2 - \cancel{\tilde{x}^2} \cancel{+1}}{\tilde{x}^2 - 1} = \frac{\tilde{x}^2}{\tilde{x}^2-1}\delta_1 + \frac{\tilde{x}^2-1}{\tilde{x}^2-1}\delta_2$$

$$= \boxed{\frac{\tilde{x}^2}{\tilde{x}^2-1}}\delta_1 + \delta_2$$

grande se $\tilde{x}^2 \approx 1$

$$n! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \ldots \cdot n \qquad f=1$$

"accomulatore"

$$\left.\begin{aligned} f &= f\cdot 1\\ f &= f\cdot 2\\ f &= f\cdot 3\\ &\vdots\\ f &= f\cdot n \end{aligned}\right] \Rightarrow$$

```
for k=1:n
  f=f*k
end
```

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$|\epsilon| < u$$

$$\frac{x^5}{5!}(1+\epsilon)$$

errore: $\boxed{\dfrac{x^5}{5!}\epsilon}$ $\qquad \dfrac{err}{\exp(x)} \leq u$

se $x \geq 0$, allora $\boxed{\dfrac{x^5}{5!} \leq \exp(x)}$ e $err \leq \exp(x)\, u$

se $x \leq 0$, $\quad 1 + (-20) + \dfrac{(-20)^2}{2} + \dfrac{(-30)^3}{3!} + \ldots$

$\quad +\quad -\quad +\quad -$

$$\frac{x^5}{5!} \cancel{\leq} \exp(x) \qquad \frac{(-20)^5}{5!} u \approx 26\cdot 000\cdot u$$

$\exp(x)$ è molto piccolo ($\approx 2\cdot 10^{-9}$ per $x=-20$)

Gli errori su ogni termine hanno coefficienti anche molto grandi: $\frac{x^{15}}{15!}$ u ≈ $2\cdot 10^7$ u

Quante operazioni?

```
for k=1:n
    y = y + pow(x,k) / factorial(k)
end
```

k op. 2 op k op.

$2k+2$ op. a ogni passo del for

$$\begin{array}{l} 2\cdot 1+2 \\ 2\cdot 2+2 \\ 2\cdot 3+2 \\ \vdots \\ 2\cdot n+2 \end{array}$$

$$1+2+3+\dots n = \frac{n(n-1)}{2}$$

$$2\cdot\frac{n(n-1)}{2} + 2\cdot n$$

operazioni

$$= O(n^2) \quad (\text{quando } n\to\infty)$$

$$= \frac{\cancel{2}n^2}{\cancel{2}} - \frac{n}{2} + 2\cdot n = n^2 + O(n)$$

[domanda: riuscireste a fare lo stesso calcolo con $O(n)$ operazione]

Indici di vettori/matrici partono da 0! ⚠

A = [1 2 ; 3 4] → matrice

v = [5 ; 6 ; 7] → vettore colonna

w = [7 8 9] → vettore riga

A(1,2) → ⚠ estende la matrice se serve

v(3)

w(1)

ones(4,5)

zeros(2,7)

eye(5)

1:2:10

size(A)

length(v) ←

length(w)

[2] [2 3] [2 3 4] ... [2 3 ... n+1]

$$\begin{bmatrix} 2 & 3 & \cdots & & n+1 \\ 3 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 3 & & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & \cdots & & & \\ 3 & \cdots & & & \\ 4 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

✓

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & & \\ & & \ddots & \ddots & \\ & & & & -1 \\ 0 & & & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} -v_1 + 2v_2 \\ \left.\begin{matrix} -v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 - v_4 \\ \vdots \\ -v_{n-2} + 2v_{n-1} - v_n \end{matrix}\right\} \text{for} \\ -v_{n-1} + 2v_n \end{bmatrix}$$

4 op per passo $\Rightarrow$ costo $O(n)$

$\sim 4n$ operazioni