METHODS FOR VERIFIED SOLUTIONS TO CONTINUOUS-TIME ALGEBRAIC RICCATI EQUATIONS

TAYYEBE HAQIRI AND FEDERICO POLONI

Abstract. We describe a procedure based on the Krawczyk method to compute a verified enclosure for the stabilizing solution of a continuous-time algebraic Riccati equation $A^*X + XA + Q = XGX$, building on the work of [B. Hashemi, SCAN 2012] and adding several modifications to the Krawczyk procedure. Moreover, we describe a new $O(n^3)$ direct method for verification, based on a fixed-point formulation of the equation inspired by the ADI procedure. The resulting methods are tested on a number of standard benchmark examples, and are competitive with the state-of-the-art methods for the same problem.

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1. Introduction

Consider a continuous-time algebraic Riccati equation *CARE*

$$A^*X + XA + Q = XGX, \tag{1.1}$$

where A, G and $Q \in \mathbb{C}^{n\times n}$ are given, G and Q are Hermitian, and $X \in \mathbb{C}^{n\times n}$ is unknown. CAREs have a variety of applications in the field of control theory and filter design, such as the linear-quadratic optimal control problem and Hamiltonian systems of differential equations [19]. A solution of (1.1) is called *stabilizing* if the *closed loop matrix* $A - GX$ is *stable*, i.e., if all its eigenvalues have strictly negative real part. If a stabilizing solution exists, it is unique [5, Theorem 2.17], and it is Hermitian, i.e., $X = X^*$. The unique stabilizing solution is the one of interest in most applications.

The solutions of (1.1) can be put in one-to-one correspondence with certain invariant subspaces of

$$H := \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix}.$$

Indeed, X is a solution of (1.1) if and only if

$$H \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} (A - GX). \tag{1.2}$$

In particular, the columns of the matrix $\begin{bmatrix} I_n \\ X \end{bmatrix}$ span an invariant subspace for the matrix H and the eigenvalues of $A - GX$ are a subset of the eigenvalues of H [5]. We refer the reader to the books by Lancaster and Rodman [19] by Bini, Iannazzo and Meini [5] for details concerning main theoretical properties and numerical solutions together with the description of the main tools for the design and analysis of solution algorithms.

The work presented in this paper addresses the problem of computing verified solutions of CAREs (1.1), that is, determining an interval matrix which is guaranteed to contain a solution of the CARE. This problem has been addressed before in the literature: the algorithms in [20] and [21], based on the interval Newton method, are pioneering works in this context but their computational complexity is $\mathcal{O}(n^6)$. In [20], the authors apply Brouwer's fixed point theorem to calculate verified solutions of the matrix Riccati equation (ARE)

$$A^T X + XA + Q = XBR^{-1}B^T X, \tag{1.3}$$

with real symmetric matrices Q and R, Q positive semi definite and R positive definite. They find an interval matrix including a positive definite solution of (1.3). The paper [30] decreases this cost to $\mathcal{O}(n^5)$ by using the Krawczyk method, which is a variant of the Newton method that does not require the inversion of an interval matrix. A major improvement is the algorithm in [13], which is applicable when the closed-loop matrix $A - GX$ is diagonalizable, and requires only $\mathcal{O}(n^3)$ operations. The recent paper [24] describes a more efficient algorithm based again on the verified diagonalization of $A - GX$. The resulting method has cubic complexity as well. An important feature of this algorithm is that does not require iteration to find a suitable candidate interval solution, unlike the previous methods based on the Krawczyk method and fixed-point theorems. This makes it faster in many cases. The same paper [24] also includes a method to verify the uniqueness and the stabilizing property of the solution.

We propose here a variant of the Krawczyk method suggested in [13], introducing several modifications. Namely:

- We use the technique introduced in [8], which consists in applying the Krawczyk method not to the original equation, but to the one obtained after a change of basis, in order to reduce the number of transformation required, with the aim to reduce the wrapping effects.
- We exploit the invariant subspace formulation (1.2) to make another change of basis in the matrix H, following a technique introduced in [22] for the *non-verified* solution of Riccati equations. This technique employs suitable permutations of H to reduce (1.1) to a different CARE whose stabilizing solution Y has bounded norm. Up to our

knowledge, this is the first attempt to use this technique in the context of interval arithmetics and verified computation.
- When applying the Krawczyk method, an enclosure interval for the so-called *slope* matrix is needed; the standard choice to compute it is using the interval evaluation of the Jacobian of the function at hand. Instead, we use a different algebraic expression which results in a smaller interval.

In addition, we present a different algorithm, based on a reformulation of (1.1) as a fixed-point equation, which achieves $O(n^3)$ and does not require the diagonalizability of the closed-loop matrix $A - GX$. This algorithm is generally less reliable than the Krawczyk-based one, but it has the advantage of not breaking down in cases in which the closed-loop matrix is defective or almost defective.

We conclude the paper by evaluating the proposed algorithms on a large set of standard benchmark problems [4, 6] for Riccati equations, comparing them with the algorithms in [13] and [24].

The paper is organized as follows. In the next section we introduce some notation and standard results in linear algebra and interval analysis which are at the basis of our methods. In section 3 we discuss various algorithms based on the Krawczyk method to compute a thin interval matrix enclosing a solution of (1.1) while in section 4.3, a fixed point approach is presented. In section 5 and 6 we perform some numerical tests and draw the conclusions and outlook, respectively.

2. Preliminaries and Notation

We try to follow the standard notation of interval analysis defined in [18]. Subsequently, we use boldface lower and upper case letters for interval scalars or vectors (boxes) and matrices, respectively, whereas lower case stands for scalar quantities and point vectors and upper case represents matrices. By $\mathbb{C}^{m\times n}(\mathbb{R}^{m\times n})$ and $\mathbb{IC}^{m\times n}_{\text{disc}}(\mathbb{IR}^{m\times n})$ we denote the sets of all complex (real) $m \times n$ matrices and the set of all complex (real) $m \times n$ interval matrices, respectively. By a complex (real) interval matrix, we mean a matrix whose entries are circular complex (compact real) intervals. For $\operatorname{mid}\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\operatorname{rad}\mathbf{A} \in \mathbb{C}^{m\times n}$, where elements in $\operatorname{rad}\mathbf{A}$ are non-negative, the notation $\mathbf{A}$ denotes the interval matrix whose center and radius are $\operatorname{mid}\mathbf{A}$ and $\operatorname{rad}\mathbf{A}$, correspondingly. Indeed, $\mathbf{A} = [\operatorname{mid}\mathbf{A} - \operatorname{rad}\mathbf{A}, \operatorname{mid}\mathbf{A} + \operatorname{rad}\mathbf{A}] := [\underline{A}, \overline{A}]$. A *Hermitian* interval matrix is an interval matrix such that both $\operatorname{mid}\mathbf{A}$ and $\operatorname{rad}\mathbf{A}$ are Hermitian; note that this does *not* imply that A is Hermitian for each $A \in \mathbf{A}$. We solve this issue by presenting an alternative method to ensure uniqueness in the following, in Section 3.5.

Complex intervals exist either as rectangles or as discs. We use here the definition as discs: a circular complex interval, or circular disc or simply a complex interval, is a closed circular disc of radius $\operatorname{rad}\mathbf{x}$ and center $\operatorname{mid}\mathbf{x}$. Indeed, it is defined as $\mathbf{x} := \{z \in \mathbb{C} : |z - \operatorname{mid}\mathbf{x}| \leq \operatorname{rad}\mathbf{x}\} = \langle \operatorname{mid}\mathbf{x}, \operatorname{rad}\mathbf{x}\rangle$. The operations on $\mathbb{IC}_{\text{disc}}$ are introduced as generalizations of operations on complex numbers. Suppose that $\circ \in \{+, -, *, /\}$ is a binary operation on the complex numbers. Then, the standard arithmetic for circular complex interval arguments $\mathbf{x} = \langle \operatorname{mid}\mathbf{x}, \operatorname{rad}\mathbf{x}\rangle$ and $\mathbf{y} = \langle \operatorname{mid}\mathbf{y}, \operatorname{rad}\mathbf{y}\rangle$ is defined as [2]:

$$\mathbf{x} \pm \mathbf{y} = \langle \operatorname{mid}\mathbf{x} \pm \operatorname{mid}\mathbf{y}, \operatorname{rad}\mathbf{x} + \operatorname{rad}\mathbf{y}\rangle\,,$$

$$\mathbf{x} * \mathbf{y} = \langle \operatorname{mid}\mathbf{x}\operatorname{mid}\mathbf{y}, |\operatorname{mid}\mathbf{x}|\operatorname{rad}\mathbf{y} + |\operatorname{mid}\mathbf{y}|\operatorname{rad}\mathbf{x} + \operatorname{rad}\mathbf{x}\operatorname{rad}\mathbf{y}\rangle\,,$$

$$1/\mathbf{x} = \langle 1/\operatorname{mid}\mathbf{x}, 1/(|\operatorname{mid}\mathbf{x}| + \operatorname{rad}\mathbf{x})\rangle \quad \text{for} \quad 0 \notin \mathbf{y},$$

$$\mathbf{x}/\mathbf{y} = \mathbf{x} * 1/\mathbf{y} \quad \text{for} \quad 0 \notin \mathbf{y}.$$

Here $|\cdot|$ denotes the absolute value of a complex number. The intersection of two interval matrices is defined component-wise where the intersection of two intervals coincides with the set theoretic definition, i.e. $\mathbf{x} \cap \mathbf{y} := \{z \in \mathbb{C} : z \in \mathbf{x} \text{ and } z \in \mathbf{y}\}$. The interval hull of two intervals $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\square(\mathbf{x}, \mathbf{y})$ which is the smallest interval containing $\mathbf{x}$ and $\mathbf{y}$.

The interval evaluation of a function $f(x)$ defined by a formula is obtained by replacing (1) the variable with an interval variable and (2) the arithmetic operations with corresponding interval operations. The result $\mathbf{f}$ is called a *natural interval extension* of f [25]. Note that, in principle, different equivalent formulas could give different interval extensions; indeed, the process of turning the customary arithmetic into interval arithmetic is not free of pitfalls; issues such as *interval dependency* and the *wrapping phenomenon* have to be considered carefully. We refer the reader to the review article [27] for a thorough introduction.

One of the basic properties of interval arithmetic, which makes its use well-founded, is that respects inclusion, i.e., it is *inclusion isotonic*. Indeed, for the four basic arithmetic operations $\circ \in \{+, -, *, /\}$ one has

$$\mathbf{x} \circ \mathbf{y} \supseteq \{x \circ y \ : \ x \in \mathbf{x},\ y \in \mathbf{y}\},$$

in which $\mathbf{x}$ and $\mathbf{y}$ are two real or circular complex intervals. For general functions defined as interval extensions, one has the following result.

Theorem 2.1. [25] *If* $\mathbf{f}$ *is an interval extension of* f *(an ordinary real or complex function of* n *variables), then*

$$f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) := \{f(x_1, x_2, \ldots, x_n) : x_1 \in \mathbf{x}_1, x_2 \in \mathbf{x}_2, \ldots, x_n \in \mathbf{x}_n\} \subseteq \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).$$

The conjugate transpose of a complex matrix A is denoted with A^*, while A^T shows the transpose of A. With the *Kronecker product* $A \otimes B$ for an $m \times n$-matrix A and a $p \times q$-matrix B we mean an $mp \times nq$ matrix defined by $A \otimes B := [a_{ij}B]$. For a point matrix $A \in \mathbb{C}^{m \times n}$, the vector $\mathrm{vec}(A) \in \mathbb{C}^{mn}$ denotes column-wise vectorization whereby the successive columns of A are stacked one below the other, beginning with the first column and ending with the last, always written as corresponding lower case. The element-wise division of the matrix $A \in \mathbb{C}^{m \times n}$ by the matrix $B \in \mathbb{C}^{m \times n}$, also known as the *Hadamard* division, denoted by $A./B$, results in the $m \times n$ matrix C whose (i, j)-th element is given by $c_{ij} = a_{ij}/b_{ij}$. For a given vector $d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{C}^n$, $\mathrm{Diag}(d) \in \mathbb{C}^{n \times n}$ is the diagonal matrix whose diagonal entries are the elements of d. Conversely, given a diagonal matrix D, $\mathrm{Diag}\, D$ is the vector whose elements are the diagonal entries of D. All of these notions and operations are analogously defined for interval quantities.

The following lemmas contain arithmetical properties of Kronecker products which we will use in the following.

Lemma 2.2. [17, 8] *Assume that A, B, C and D be complex matrices with compatible sizes and $a = \mathrm{vec}\, A$ and $b = \mathrm{vec}\, B$. Then,*

(1) $(A \otimes B)(C \otimes D) = AC \otimes BD$,
(2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
(3) $(A \otimes B)^* = A^* \otimes B^*$,
(4) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,
(5) $\mathrm{vec}(ABC) = (C^T \otimes A)b$,
(6) $(\mathrm{Diag}(a))^{-1}b = b./a$.

Lemma 2.3. [8] *Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be complex interval matrices of compatible sizes and $\mathbf{a} = \mathrm{vec}\, \mathbf{A}$ and $\mathbf{b} = \mathrm{vec}\, \mathbf{B}$. . Then,*

$$a)\Big\{(C^T \otimes A)b \ : \ A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C}\Big\} \subseteq \begin{cases} \mathrm{vec}\Big(\mathbf{A}(\mathbf{BC})\Big), \\ \mathrm{vec}\Big((\mathbf{AB})\mathbf{C}\Big), \end{cases}$$

$$b)\Big(\mathrm{Diag}(\mathbf{a})\Big)^{-1}\mathbf{b} = \mathbf{b}./\mathbf{a}.$$

The CPU rounding mode called *round to nearest* is typically used when we compute using a computer. Other rounding modes are *round downward* (round towards the largest floating point number that is smaller than the real result), and *round upward* (round towards the smallest number larger than the real result) etc. [1]. In general, it is impossible to compute the exact solution in a numerical computation using floating point numbers. Then, a numerical computation with guaranteed accuracy switches the CPU rounding mode, computes a lower and an upper bound to the true solution, and creates an interval which is guaranteed to contain it [12]. One example of software

which provides a fast implementation of such a reliable interval arithmetic is the MATLAB toolbox INTLAB [28]; older versions of INTLAB are freely available for noncommercial use. The default arithmetic for both real and complex intervals in INTLAB is the midpoint-radius arithmetic [27].

3. Finding enclosures for the solutions to CAREs based on Krawczyk's method

Enclosure methods using interval arithmetic work this way: let $g : \mathbb{C}^n \to \mathbb{C}^n$ be some function of which we wish to find a zero. Then the mathematical algorithm would, for example, be based on a fixed point operator $h : \mathbb{C}^n \to \mathbb{C}^n$ whose fixed points are known to be the zeros of g. Assume that h is continuous and that we know an enclosure function $\mathbf{h}$ for h, i.e., a function based on correct interval arithmetic which gives an interval vector $\mathbf{h}(\mathbf{x})$ containing the range of $\mathbf{h}$ over a given interval vector $\mathbf{x}$. Then if $\mathbf{h}(\mathbf{x}) \subseteq \mathbf{x}$ we know that $h(\mathbf{x}) \subseteq \mathbf{x}$ and so h has a fixed point in $\mathbf{x}$ by Brouwer's theorem [7].

In this paper, often the functions $\mathbf{h}$ are variants of the Krawczyk operator. To define this operator, we first need the concept of a slope.

Definition 3.1. [25] Suppose $f : \psi \subseteq \mathbb{C}^N \to \mathbb{C}^N$ and $x, y \in \mathbb{C}^N$. Then, the *slope* $S(f; x, y)$ is defined to be that mapping such that

$$f(y) - f(x) = S(f; x, y)(y - x).$$

Note that $S(f; x, y)$ exists for particular f, x and y under various assumptions.

We are now ready to state the result which is at the basis of all the modified Krawczyk-type algorithms used in the rest of our paper.

Theorem 3.2. [10] *Assume that $f : \psi \subset \mathbb{C}^N \to \mathbb{C}^N$ is continuous. Let $\tilde{x} \in \psi$ and $\mathbf{z} \in \mathbb{IC}^N$ be such that $\tilde{x} + \mathbf{z} \subset \psi$. Moreover, assume that $\mathcal{S} \subset \mathbb{C}^{N\times N}$ is a set of matrices containing all slopes $S(f; \tilde{x}, y)$ for $y \in \tilde{x} + \mathbf{z} =: \mathbf{x}$. Finally, let $R \in \mathbb{C}^{N\times N}$. Denote by $\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S})$ the set*

$$\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) := \{-Rf(\tilde{x}) + (I - RS)z \ : \ S \in \mathcal{S}, \ z \in \mathbf{z}\}.$$

If

$$\mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \mathrm{int}(\mathbf{z}), \tag{3.1}$$

in which int$(\mathbf{z})$ is the topological interior of $\mathbf{z}$, then the function f has a zero x^* in $\tilde{x} + \mathcal{K}_f(\tilde{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \mathbf{x}$.

Moreover, if $\mathcal{S}$ also contains all slope matrices $S(f; x, y)$ for $x, y \in \mathbf{x}$, then x^* is the only zero of f contained in $\mathbf{x}$.

The standard Krawczyk operator

$$\mathbf{k}_f(\tilde{x}, R, \mathbf{z}, \mathbf{S}) := -Rf(\tilde{x}) + (I - R\mathbf{S})\mathbf{z}, \tag{3.2}$$

may be use for a computational existence test, where $\mathbf{S}$ is an interval matrix containing all slopes $S(f;x,y)$ for $x,y \in \mathbf{x}$; the standard choice is $\mathbf{S} = \mathbf{f}'(\mathbf{x})$, the interval arithmetic evaluation of the Jacobian $f'(x)$. By the enclosure property of interval arithmetic,

$$\mathbf{k}_f(\tilde{x}, R, \mathbf{z}, \mathbf{S}) \subset \operatorname{int} \mathbf{z} \tag{3.3}$$

implies (3.1). So, if (3.3) is satisfied then f has a zero in $\tilde{x}+\mathbf{k}_f(\tilde{x}, R, \mathbf{z}, \mathbf{S})$. The crucial relation (3.3) is plausible in computation if the two terms $-Rf(\tilde{x})$ and $I - R\mathbf{S}$ in the standard Krawczyk operator above are close to the zero interval vector and the zero interval matrix, respectively. This means that we have to take as $\tilde{x}$ a good approximation of a zero of f and as R a good approximation of $(f'(\tilde{x}))^{-1}$, both obtained via a classic floating point algorithm.

3.1. **A modified Krawczyk algorithm approach.** We now introduce the concepts that are needed to apply the modified Krawczyk method to solve a matrix equation such as (1.1). The *Fréchet derivative* [16] of a matrix-valued function $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ at a point $X \in \mathbb{C}^{n\times n}$ is a linear mapping $L_X : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that for all $E \in \mathbb{C}^{n\times n}$

$$F(X+E) - F(X) - L_X(E) = o(\|E\|).$$

Since L_X is a linear operator, we can write

$$\operatorname{vec}(L_X(E)) = K_X \operatorname{vec}(E),$$

for a matrix $K_X \in \mathbb{C}^{n^2\times n^2}$ that dependents on L but not E. One refers to K_X as the *Kronecker form* of the Fréchet derivative.

In the case of the continuous-time algebraic Riccati equation (1.1), the function is $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ defined as

$$F(X) := A^*X + XA + Q - XGX.$$

For this function, one has

$$L_X(E) = E(A - GX) + (A - GX)^*E,$$

and its Kronecker form is

$$K_X = I \otimes (A - GX)^* + (A - GX)^T \otimes I. \tag{3.4}$$

We wish to use the modified Krawczyk algorithm on the function obtained by regarding F as a vector map $f : \mathbb{C}^N \to \mathbb{C}^N$, with $N = n^2$, defined by

$$f(x) := \operatorname{vec}(A^*X + XA + Q - XGX), \quad x = \operatorname{vec} X. \tag{3.5}$$

The following result, which appeared in [13], shows that the interval version of the Kronecker form of the Fréchet derivative can be used as an enclosure for the slope in the modified Krawczyk method.

Theorem 3.3. [13] *Let* $\mathbf{X}$ *be a Hermitian interval matrix, and* $\mathbf{K_X} = I \otimes (A - G\mathbf{X})^* + (A - G\mathbf{X})^T \otimes I$ *be the natural interval extension of* K_X *in* (3.4). *Then for each* $X, Y \in \mathbf{X}$, *it holds that* $S(f; x, y) \in \mathbf{K_X}$ *in which* $x = \operatorname{vec} X$ *and* $y = \operatorname{vec} Y$.

The next ingredient that we need to apply the Krawczyk algorithm is the matrix R. One would like to use $R = K_{\check{X}}^{-1}$, where $\check{X}$ is a solution to the CARE (1.1) computed in floating point arithmetic. However, this is the inverse of an $n^2 \times n^2$ matrix, whose computation costs would $O(n^6)$ floating point operations. Even considering the Kronecker product structure of $K_{\check{X}}$, there is no algorithm in literature to compute R explicitly with less than $O(n^5)$ arithmetic operations. The action of R, that is, computing the product Rv given a vector $v \in \mathbb{C}^N$, can be computed with $O(n^3)$ operations with methods such as the Bartels-Stewart algorithm [3]. However, this method cannot be used effectively in conjunction with interval arithmetic due to excessive wrapping effects, as argued in [9].

The work [13] (and, earlier, on a similar function, [8]) contains an alternative method to perform this computation with complexity $O(n^3)$, in the case when $A - G\check{X}$ is diagonalizable. Assume that an approximate eigendecomposition of $A - G\check{X}$ is available, that is,

$$A - G\check{X} \approx V \Lambda W \quad \text{with } V, W, \Lambda \in \mathbb{C}^{n \times n}, \tag{3.6a}$$

$$\Lambda \approx \operatorname{Diag}(\lambda_1, \dots, \lambda_n), \quad VW \approx I. \tag{3.6b}$$

We write $\approx$ instead of $=$ because V, $W \approx V^{-1}$ and λ_i, $i = 1, \dots, n$ are computed numerically with a standard method such as Matlab's `eig`. So, equality does not hold (in general) in the mathematical sense. Once these quantities are computed, we can choose R as

$$R = (V^{-T} \otimes W^*) \cdot \Delta^{-1} \cdot (V^T \otimes W^{-*}),$$

where $\Delta = I \otimes \Lambda^* + \Lambda^T \otimes I$. Then, $R \approx K_{\check{X}}^{-1}$ holds.

For the numerical computation of a superset of $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{S})$, we use interval arithmetic to evaluate the Krawczyk operator $\mathbf{k}_f(\check{x}, R, \mathbf{z}, \mathbf{S})$, where $\mathbf{S} \supseteq \mathcal{S}$ is an interval matrix containing all slopes $S(f; x, y)$ for $x, y \in \mathbf{x} := \check{x} + \mathbf{z}$. If we manage to find an interval $\mathbf{z}$ such that

$$\mathbf{k}_f(\check{x}, R, \mathbf{z}, \mathbf{S}) \subset \operatorname{int}(\mathbf{z}),$$

then, by the enclosure property of interval arithmetics, (3.1) holds, and this in turn implies that f has a zero in $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{S})$.

The standard method [27] to obtain such an interval $\mathbf{z}$ is an iterative one. We start from the residual matrix $\mathbf{z}_0 := \mathbf{F}(\check{\mathbf{X}})$, and proceed alternating successive steps of enlarging this interval with a technique known as *ε-inflation* [27], applying the Krawczyk operator to it, $\mathbf{z}_{i+1} := \mathbf{k}_f(\check{x}, R, \mathbf{z}_i, \mathbf{S})$, and intersecting

the interval obtained at successive iterations. This is ultimately a trial-and-error procedure, which is not guaranteed to succeed: the operator $\mathbf{k}_f$ may simply not contract its interval argument $\mathbf{z}$ sufficiently. This may be due to ill-conditioning of the original equation, to a bad choice of R, or to the excessive growth of the intervals in the numerical computations (wrapping effect).

One can see the strategy that we employ in our experiments in Algorithm 1. It is a version with two attempts at proving inclusion at every iteration of the loop, which appeared also in the works [8, 14, 15].

Notice the ε-inflation, which is performed by adding $\texttt{realmin} \cdot [-1, 1]$ to the computed interval. Here `realmin` denotes the smallest positive normalized floating point number.

3.2. **Affine transform enclosure.** The main difficulty in using interval arithmetic to verify the existence of a solution is the so-called *wrapping effect* [27]: given interval matrices $\mathbf{A}$ and $\mathbf{B}$, the set $\{AB\colon A \in \mathbf{A}, B \in \mathbf{B}\}$ is not (in general) an interval matrix, and to represent it in interval form we have to enlarge it by replacing with an enclosing interval. This effect is more pronounced in presence of repeated successive multiplications, intervals with large diameter, and ill-conditioned matrices. This increase may prevent us to verify computationally the critical condition (3.1).

Reducing the impact of the wrapping effect can give a reduction in the diameter of the computed solution interval and also in the computational time, since it can reduce the number of iterations needed before a successful inclusion is computed.

In this section we describe a technique for reducing the wrapping effect in the modified Krawczyk method, which has already been successfully applied to several matrix equations [8, 10]. The main idea is applying the verification algorithm to a modified function $\hat{f}$ obtained from f via an affine transformation; in this way, we reduce the number of interval operations to perform inside the verification procedure.

We define the function

$$\hat{f}(\hat{x}) := (V^T \otimes W^{-*}) f((V^{-T} \otimes W^*)\hat{x}).$$

If $\check{x} = \operatorname{vec} \check{X}$ is an approximate solution to $f(x) = 0$, then $\hat{\check{x}} = (V^T \otimes W^{-*})\check{x}$ is an approximate solution to $\hat{f}(\hat{x}) = 0$. A set of slopes for $\hat{f}$ can be defined as

$$\hat{\mathcal{S}} := \{\hat{S}(\hat{f}; \hat{x}, \hat{y}),\ \hat{x}, \hat{y} \in \hat{\mathbf{x}} := \hat{\check{x}} + \hat{\mathbf{z}}\}.$$

Defining $x = (V^{-T} \otimes W^*)\hat{x}$, $y = (V^{-T} \otimes W^*)\hat{y}$, we have

$$\begin{aligned}\hat{S}(\hat{f};\hat{x},\hat{y})(\hat{x}-\hat{y}) &= \hat{f}(\hat{x}) - \hat{f}(\hat{y}) \\ &= (V^T \otimes W^{-*})(f(x) - f(y)) \\ &= (V^T \otimes W^{-*})S(f;x,y)(x-y) \\ &= (V^T \otimes W^{-*})S(f;x,y)(V^{-T} \otimes W^*)(\hat{x}-\hat{y}).\end{aligned}$$

Hence

$$\hat{S}(\hat{f};\hat{x},\hat{y}) = (V^T \otimes W^{-*})S(f;x,y)(V^{-T} \otimes W^*). \tag{3.7}$$

In particular, if we combine this result with Theorem 3.3, we find that we can take in the modified Krawczyk method

$$\hat{\mathcal{S}} = I \otimes (W(A - G\mathbf{X})W^{-1})^* + (V^{-1}(A - G\mathbf{X})V)^T \otimes I. \tag{3.8}$$

Observe that

$$I \otimes (W(A - G\check{X})W^{-1})^* + (V^{-1}(A - G\check{X})V)^T \otimes I \approx I \otimes \Lambda^* + \Lambda^T \otimes I,$$

so a natural choice for $\hat{R}$ is the diagonal matrix

$$\hat{R} = \Delta^{-1}, \quad \Delta := I \otimes \Lambda^* + \Lambda^T \otimes I.$$

Now, we compute an enclosure for $\mathcal{K}_{\hat{f}}(\hat{\check{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathcal{S}})$ which can be written as $\mathbf{k}_{\hat{f}}(\hat{\check{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathbf{S}})$ in which $\check{x}$ is an approximate solution for (1.1), $\hat{R}$ is Δ^{-1}, $\hat{\mathbf{Z}} = W^{-*}\mathbf{Z}V$ and $\hat{\mathcal{S}} = \{\hat{S}(\hat{f};\hat{x},\hat{y}),\ \hat{x},\hat{y} \in \hat{\mathbf{x}} := (V^T \otimes W^{-*})\check{x} + \hat{\mathbf{z}}\}$. We also take care that the quantities which are not available exactly are enclosed into computable quantities in interval forms, for instance I_V and I_W are interval matrices which are known to contain the exact value of V^{-1} and W^{-1}, appropriately. More details for computing the superset

$$\begin{aligned}\mathbf{k}_{\hat{f}}(\hat{\check{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathbf{S}}) &= -\hat{R}\hat{f}(\hat{\check{x}}) + (I - \hat{R}\hat{\mathbf{S}})\hat{\mathbf{z}} \\ &= -\Delta^{-1}((V^T \otimes W^{-*})f(\check{x}) \\ &\quad - (\Delta - I \otimes (W(A - G\mathbf{X})W^{-1})^* \\ &\quad - (V^{-1}(A - G\mathbf{X})V)^T \otimes I))\hat{\mathbf{z}}\end{aligned}$$

for $\mathcal{K}_{\hat{f}}(\hat{\check{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathcal{S}}) := \{-\hat{R}\hat{f}(\hat{\check{x}}) + (I - \hat{R}\hat{S})\hat{z} \ : \ \hat{S} \in \hat{\mathcal{S}}, \hat{z} \in \hat{\mathbf{z}}\}$, are displayed in Algorithm 2. The complete algorithm is shown in Algorithm 1. In all algorithms, whenever the evaluation order of an expression is not specified exactly due to missing brackets, we evaluate from left to right.

An important observation is that the last transformation $\mathbf{X} = \check{X} + W^*\hat{\mathbf{Z}}\mathbf{I}_V$ happens after the Krawczyk verification procedure. So, while the procedure guarantees that only one zero x^* of $\hat{f}$ is contained in $W^{-*}\check{X}V + \hat{\mathbf{Z}}$, when we return to the original setting and compute an enclosure for $\mathbf{X} = \check{X} + W^*\hat{\mathbf{Z}}\mathbf{I}_V$,

other solutions of (1.1) may fall into this enclosure. Hence, this variant of the modified Krawczyk method does *not* guarantee that there is a unique solution of (1.1) in $\mathbf{X}$.

On the other hand, this variant reduces the number of $n \times n$ matrix multiplications to perform inside the verification loop from 12 to 10, and this can be useful in reducing the wrapping effects.

3.3. **Verifying a different Riccati equation.** Another possible modification to the verification process consists in modifying the equation into one with (possibly) better numerical properties. The idea stems from the formulation (1.2) of a CARE as an invariant subspace problem. We start from the following result.

Lemma 3.4. *The stabilizing solution X of CARE (1.1) is the only matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$H \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} R, \quad H = \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} \in \mathbb{C}^{2n\times 2n} \tag{3.9}$$

for some stable matrix R. Moreover, it holds that $R = A - GX$.

The subspace $\operatorname{im} \begin{bmatrix} I_n \\ X \end{bmatrix}$ is called *stable invariant subspace* of the matrix H. We use this formulation to relate the solution X to the one of a different CARE.

Lemma 3.5. *Let X be the stabilizing solution of (1.1). Suppose that $P \in \mathbb{C}^{2n\times 2n}$ be a matrix such that $P^{-1}HP$ has the same structure as H, i.e.,*

$$P^{-1}HP = \begin{bmatrix} A_P & -G_P \\ -Q_P & -A_P^* \end{bmatrix} \tag{3.10}$$

for some matrices $A_P, G_P = G_P^*, Q_P = Q_P^* \in \mathbb{C}^{n\times n}$. Let Y be the stabilizing solution of the CARE

$$A_P^* Y + Y A_P + Q_P = Y G_P Y \tag{3.11}$$

and $U_1, U_2 \in \mathbb{C}^{n\times n}$ be defined by

$$P \begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

If U_1 is invertible, then $X = U_2 U_1^{-1}$.

Proof. We have

$$P^{-1}HP \begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} I_n \\ Y \end{bmatrix} R_P$$

for a stable matrix R_P. Multiplying both sides by P on the left we get

$$H \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} R_P,$$

and then multiplying on the right by U_1^{-1}

$$H \begin{bmatrix} I_n \\ U_2 U_1^{-1} \end{bmatrix} = \begin{bmatrix} I_n \\ U_2 U_1^{-1} \end{bmatrix} U_1 R_P U_1^{-1}.$$

Since $U_1 R_P U_1^{-1}$ is stable, Lemma 3.4 gives us the thesis. □

The paper [22] contains a convenient strategy to construct a matrix P with a particularly simple form (a permutation matrix with some sign changes) for which all the required assumptions hold and in addition Y is bounded. Define for each $k = 1, 2, \dots, n$

$$S_k = \left[\begin{array}{ccc|ccc} I_{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I_{n-k} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{k-1} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-k} \end{array}\right] \in \mathbb{C}^{2n \times 2n},$$

i.e., the matrix whose action is swapping the entries k and $n+k$ of a vector, changing sign to one of them. The matrices S_k are orthogonal and commute with each other.

Theorem 3.6. [22, Theorem 3.4] *Let $\mathcal{I} = \{i_1, i_2, \dots, i_k\}$ be a subset of $\{1, 2, \dots, n\}$, and $P = S_{i_1} S_{i_2} \cdots S_{i_k}$.*

1. *For each choice of $\mathcal{I}$, the matrix $P^{-1} H P$ has the structure (3.10).*
2. *For each $\tau \geq \sqrt{2}$, one can find $\mathcal{I}$ such that U_1 is nonsingular and Y has all its elements bounded in modulus by τ (referring to the definitions of U_1 and Y in Lemma 3.5).*

Since Y is bounded, we expect the numerical solution of the CARE (3.11) to have better numerical properties than the one of the original CARE (1.1), especially in cases where X has large norm. This method suggests an alternative verification strategy:

1. Compute P satisfying Theorem 3.6. The MATLAB toolbox [26] contains an algorithm to do it.
2. Form the coefficients A_P, G_P and Q_P, which can be obtained from the entries of H only using permutations and sign changes.
3. Using one of the various verification methods for CAREs, compute an interval $\mathbf{Y}$ containing Y.

(4) Compute
$$\begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix} = P\begin{bmatrix}I\\ \mathbf{Y}\end{bmatrix},$$
which, again, requires only rearranging the entries and changing their signs, and hence can be done without wrapping effects.
(5) Compute using interval arithmetic $\mathbf{X} = \mathbf{U}_2\mathbf{U}_1^{-1}$.

Then, clearly, $\mathbf{X}$ contains the true solution X^* of (1.1). Again, since the matrix $\mathbf{X}$ computed in the last step is only an enclosure for $\mathbf{U}_2\mathbf{U}_1^{-1}$, it might be the case that other solutions of the CARE (1.1) are contained in $\mathbf{X}$ in addition to X^*.

3.4. **A new superset.** According to Theorem 3.2, the computed interval is guaranteed to contain a unique solution if the set $\mathbf{S}$ contains the slopes $S(f; x, y)$ for all $x, y \in \mathbf{x}$. On the other hand, if we employ an interval matrix containing only the slopes $S(f; \check{x}, y)$ for all $y \in \mathbf{x}$, existence can be proved, but not uniqueness. Since we have already decided to forgo (for now) uniqueness by introducing the improvements in the two previous subsections, it makes sense to let go of it also when choosing the superset $\mathbf{S}$. We show here how to compute a smaller interval matrix which still satisfies the requirements for the existence of the solution in Theorem 3.2.

Theorem 3.7. *Let f be as in (3.5), $\mathbf{X} \in \mathbb{IC}^{n\times n}$ be an interval matrix, and $\check{X} \in \mathbf{X}$ be Hermitian. Then, the interval matrix*
$$I \otimes (A - G\check{X})^* + (A - G\mathbf{X})^T \otimes I$$
contains the slopes $S(f, \check{x}, y)$ for each $Y \in \mathbf{X}$ where $\check{x} = \operatorname{vec}\check{X}$ and $y = \operatorname{vec} Y$.

Proof. We have
$$\begin{aligned} f(\check{x}) - f(y) &\\ &= \operatorname{vec}(A^*\check{X} + \check{X}A + Q - \check{X}G\check{X} - A^*Y - YA - Q + YGY)\\ &= \operatorname{vec}\left((A^* - \check{X}G)(\check{X} - Y) + (\check{X} - Y)(A - GY)\right)\\ &= (I \otimes (A - G\check{X})^* + (A - GY)^T \otimes I)(\check{x} - y), \end{aligned}$$
hence $S(f; \check{x}, y) = (I \otimes (A - G\check{X})^* + (A - GY)^T \otimes I)$. □

Note that we can write $A^* - \check{X}G = (A - G\check{X})^*$ because $\check{X}$ is symmetric. One could obtain the equivalent expression $I \otimes (A - G\mathbf{X})^* + (A - G\check{X})^T \otimes I$ with a similar proof, but this form would require the hypothesis that $\mathbf{X}$ is a symmetric interval.

As a consequence of Theorem 3.7, we can replace (3.8) with
$$\hat{\mathcal{S}} = I \otimes (W(A - G\check{X})W^{-1})^* + (V^{-1}(A - G\mathbf{X})V)^T \otimes I \tag{3.12}$$

in our modified Krawczyk algorithm applied to $\hat{f}$, and it will still yield an interval matrix containing a (possibly non-unique) solution of (1.1).

Algorithm 1 Computation of an interval matrix $\mathbf{X}$ containing at least one solution of CARE (1.1)

1: Compute an approximate stabilizing solution $\check{X}$ of CARE (1.1) in floating point;
2: Compute approximations V, W, Λ for the eigendecomposition of $A - G\check{X}$ in floating point; {For instance, using the MATLAB command `eig`};
3: Compute $D = \overline{\mathrm{Diag}(\Lambda)}[1, 1, \ldots, 1]_{1\times n} + [1, 1, \ldots, 1]^T_{1\times n}(\mathrm{Diag}(\Lambda))^T$;
4: Compute interval matrices $\mathbf{I}_V$ and $\mathbf{I}_W$ containing V^{-1} and W^{-1}, resp; {For instance, take `verifylss.m` from INTLAB};
5: Compute the interval matrix $\hat{\mathbf{G}}$ with $-\hat{R}\hat{f}(\hat{x}) \in \hat{\mathbf{G}}$; where $\hat{\mathbf{G}}$ is obtained from line 2-4 in Algorithm 2;
6: Put $k = 0$ and $\hat{\mathbf{Z}} = \hat{\mathbf{G}}$;
7: **for** $k = 1, \ldots, k_{max}$ **do**
8: Put $\hat{\mathbf{Y}} := \square(0, \hat{\mathbf{Z}} \cdot \texttt{infsup}(0.9, 1.1) + \texttt{realmin} \cdot [-1, 1]$;{$\epsilon-$inflation technique}
9: Compute $\hat{\mathbf{N}}$ using $\check{X}$, $\hat{\mathbf{Y}}$ from line 5-9 in Algorithm 2;
10: **if** $\hat{\mathbf{K}} = \hat{\mathbf{G}} + \hat{\mathbf{N}} \subset \mathrm{int}\hat{\mathbf{Y}}$ {successful inclusion} **then**
11: $\hat{\mathbf{Z}} = \hat{\mathbf{K}}$;
12: break;
13: **end if**
14: $\hat{\hat{\mathbf{Z}}} = \hat{\mathbf{Y}} \cap \hat{\mathbf{K}}$;
15: Compute $\hat{\hat{\mathbf{N}}}$ as in Algorithm 2 using $\hat{\hat{\mathbf{Z}}}$ instead of $\hat{\mathbf{Y}}$;
16: **if** $\hat{\hat{\mathbf{K}}} = \hat{\mathbf{G}} + \hat{\hat{\mathbf{N}}} \subset \mathrm{int}\hat{\hat{\mathbf{Z}}}$ {successful inclusion} **then**
17: $\hat{\mathbf{Z}} = \hat{\hat{\mathbf{K}}}$;
18: break;
19: **end if**
20: $\hat{\mathbf{Z}} = \hat{\mathbf{Z}} \cap \hat{\hat{\mathbf{K}}}$;
21: **end for**
22: $\mathbf{X} := \check{X} + W^* \hat{\mathbf{Z}} \mathbf{I}_V$
23: Output $\mathbf{X}$.

Let us precisely analyze the computational complexity of Algorithm 3.

Theorem 3.8. *The time complexity of Algorithm 3 is $\mathcal{O}(n^3)$ floating point operations.*

Proof. Computing $\check{X}$ in Line 1 requires $\mathcal{O}(n^3)$ operations, using for instance the algorithm mentioned in [24] (based on the ordered Schur form of H and an

Algorithm 2 Enclosing $\mathcal{K}_{\hat{f}}(\hat{\tilde{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathcal{S}})$ by evaluating $\mathbf{k}_{\hat{f}}(\hat{\tilde{x}}, \hat{R}, \hat{\mathbf{z}}, \hat{\mathbf{S}})$

Input A, G, Q, $\check{X}$, $\hat{\mathbf{Z}}$;
{In this function we rely on $V, W, \mathbf{I}_V, \mathbf{I}_W, D$ already computed in Algorithm 1}
$\mathbf{F} = Q + \check{X}A + A^*\check{X} - \check{X}G\check{X}$;
$\hat{\mathbf{F}} = \mathbf{I}_W^* \mathbf{F} V$;
$\hat{\mathbf{G}} = -\hat{\mathbf{F}}./D$;
$\hat{\mathbf{Y}} = W^* \hat{\mathbf{Z}} \mathbf{I}_V$;
$\hat{\mathbf{P}} = \mathbf{I}_W^*(A - G\check{X})^* W^*$; {modified superset; see Section 3.4}
$\hat{\mathbf{Q}} = \mathbf{I}_V(A - G(\check{X} + \hat{\mathbf{Y}}))V$;
$\hat{\mathbf{E}} = (\Lambda^* - \hat{\mathbf{P}})\hat{\mathbf{Y}} + \hat{\mathbf{Y}}(\Lambda - \hat{\mathbf{Q}})$;
$\hat{\mathbf{N}} = \hat{\mathbf{E}}./D$;
$\hat{\mathbf{K}} = \hat{\mathbf{G}} + \hat{\mathbf{N}}$;
Output $\hat{\mathbf{K}}$.

Algorithm 3 Verified computation of solution of CARE (1.1) using Lagrangian permuted graph bases

Input A, G, Q;
Compute an approximate solution $\check{X}$ of (1.1) in floating point;
Compute a matrix P satisfying point 2 of Theorem 3.6 {For instance with the toolbox [26]};
Compute A_p, G_p, Q_p satisfying $PHP^T = [A_p \ -G_p; -Q_p \ -A_p]$;
Compute a verified solution $\mathbf{Y}$ to (3.11) by using Algorithm 1 or Algorithm 5;
Set $\begin{bmatrix} \mathbf{U_1} \\ \mathbf{U2} \end{bmatrix} = P \begin{bmatrix} \mathbf{I} \\ \mathbf{Y} \end{bmatrix}$;
Compute $\mathbf{X} = \mathbf{U_2}{\mathbf{U_1}}^{-1}$;
Output $\mathbf{X}$.

additional Newton step with the residual computation performed in emulated quadruple-precision arithmetic). Forming P in Theorem 3.6 via the approach explored in [26] costs n^3 floating point operations. Computing $\mathbf{Y}$ by using Algorithm 1 has also cost $\mathcal{O}(n^3)$ (assuming that a moderate number of steps is sufficient). The cost for the eigendecomposition and the enclosures $\mathbf{I}_V$ and $\mathbf{I}_W$ is again cubic in n. All the other matrix-matrix operations (including the Hadamard divisions) in Algorithms 1 and 2 have again cost $\mathcal{O}(n^3)$ at most, as they only involve $n \times n$ matrices. The total cost is therefore $\mathcal{O}(n^3)$. □

3.5. **Uniqueness issues.** As noted before, the modifications to the Krawczyk method introduced here do not ensure that the found interval matrix contains only one solution to (1.1), and hence in principle it is not even guaranteed

that the computed solution is the stabilizing one. However, we know that if a CARE has a stabilizing solution, then it is unique [5, Theorem 2.17]: this is a consequence of the invariant subspace formulation (1.2) and of the symmetry in the eigenvalue of the Hamiltonian [5, Section 1.5]. Hence, if all matrices contained in $A - G\mathbf{X}$ are stable, then the solution $X \in \mathbf{X}$ is verified to be stabilizing *and* unique. To verify stabilizability, we can use the method described in [23], which is summarized in [24, Lemma 2.4]. The resulting method is described in Algorithm 4. In the algorithm, we use the notation $\Re z$ to mean the real part of the complex number z.

Algorithm 4 Verifying the stabilizing property of an interval matrix $\mathbf{X}$

Input A, G, $\mathbf{X}$;
Compute approximations V, W, Λ for the eigendecomposition of $\operatorname{mid}(A - G\mathbf{X})$ in floating point; {For instance, using the MATLAB command `eig`};
$R = \texttt{mag}(W((A - G\mathbf{X})V - \texttt{intval}(V)\Lambda))$;
$S = \texttt{mag}(I_n - \texttt{intval}(V)W)$;
$e = [1, 1, \dots, 1]^T_{1\times n}$;
$u = Re$;
$t = Se$;
$\mu = \max(u./(e - t))$;
$r = u + \mu t$;
if $\max(t) < 1$ and $r + \max(\Re(\operatorname{Diag}\Lambda)) < 0$ **then**
    return `success`;
else
    return `failure to verify stabilizability`;
end if

Notice one subtle point: in Algorithm 4 we compute V, Λ and W from the eigendecomposition of $\operatorname{mid}(A - G\mathbf{X})$; this differs slightly from using the value of V computed previously, which was the eigenvector matrix of $A - G\check{X}$. Now that the inclusion interval $\mathbf{X}$ is known, this choice gives better results in our experiments. The cost for this verification is again $O(n^3)$ floating point operations.

4. A direct fixed-point method

While the methods described in the previous sections work for many examples of Riccati equations, an essential limitation is that all of them require the closed-loop matrix $A - G\check{X}$ to be diagonalizable. Products with the eigenvector matrix V and its inverse are required along the algorithm, and if these are ill-conditioned then the wrapping effects are more pronounced and the required inclusion is less likely to hold. A striking example of this phenomenon is the first example in the benchmark set [4]. This is a simple 2×2 problem

which appears in [4] as nothing more than a "warm-up example", and yet all the verification methods described here (including those in [13] and [24]) fail, since $A - G\check{X}$ is not diagonalizable.

To solve this issue, we would like to propose a different method for verification. The procedure is based on some ideas which appear in the context of ADI methods [29]. While this method is somehow more primitive and works on a lower number of examples, it does not require that the closed-loop matrix be diagonalizable.

We rewrite the CARE (1.1) as follows. Given any $\check{X} \in \mathbb{C}$, one can write the stabilizing solution X as $X = \check{X} + Z$ for an unknown correction matrix Z, and rewrite (1.1) as a Riccati equation in Z,

$$\tilde{A}^* Z + Z\tilde{A} + \tilde{Q} = ZGZ, \quad \tilde{A} = A - G\check{X}, \quad \tilde{Q} = A\check{X} + \check{X}A + Q - \check{X}G\check{X}. \tag{4.1}$$

Note that the degree-two coefficient G is unchanged. The stabilizing solution of this equation is Z, since $\tilde{A} - GZ = A - GX$ is stable. For any $s \in \mathbb{C}$ such that $\tilde{A} - sI$ is nonsingular, (4.1) is equivalent to the fixed point equation

$$Z = (\tilde{A} - sI)^{-*}(ZGZ - \tilde{Q} - Z(\tilde{A} + sI)).$$

Thus, if we find an interval $\mathbf{Z}$ such that $(\tilde{A} - sI)^{-*}(\mathbf{Z}G\mathbf{Z} - \tilde{Q} - \mathbf{Z}(\tilde{A} + sI)) \subseteq \mathbf{Z}$, it follows from the Brouwer fixed-point theorem that (4.1) has a solution $Z \in \mathbf{Z}$, and that (1.1) has a solution such as $X \in \check{X} + \mathbf{Z}$.

This simple iterative method is effective when $(\tilde{A} - sI)^{-*}\mathbf{Z}(\tilde{A} + sI)$ does not suffer excessively from wrapping effects, since we can expect $\tilde{Q}$ and the quadratic term $\mathbf{Z}G\mathbf{Z}$ to be small.

Are there any preconditioning transformations that we can make to improve the method? A possibility is applying a change of basis to the whole problem. Let $V \in \mathbb{C}^{n\times n}$ be invertible; we set

$$Z_V = V^* Z V, \quad A_V = V^{-1}\tilde{A}V, \quad Q_V = V^*\tilde{Q}V, \quad G_V = V^{-1}GV^{-*},$$

so that (4.1) is transformed into

$$A_V^* Z_V + Z_V A_V + Q_V = Z_V G_V Z_V. \tag{4.2}$$

Continuing as above, we obtain the fixed-point equation

$$Z_V = (A_V - sI)^{-*}(Z_V G_V Z_V - Q_V - Z_V(A_V + sI)). \tag{4.3}$$

If $\tilde{A}$ is diagonalizable, we can set V as its computed approximate eigenvalue matrix, as in (3.6). One can see then that the resulting method has several steps in common with the Krawcyk method described in the previous sections.

We have more freedom, though: this time, there is no risk of running into $O(n^6)$ operations, since everything in (4.3) is computable explicitly with standard linear algebra operations. It is sufficient to choose a matrix V such that the off-diagonal part of A_V is small with respect to its diagonal.

Some heuristic experimentation leads us to the following choices: we take $s = -\min\{\Re\lambda : \lambda \text{ is an eigenvalue of } \operatorname{mid}\tilde{\mathbf{A}}\}$ (motivated by the idea to make $A_V + sI$ small and $A_V - sI$ large), and V as the orthogonal matrix such that $V^{-1}\tilde{A}V$ is upper triangular (Schur factorization), motivated by the idea to concentrate most of the norm of $V^{-1}\tilde{A}V$ on its diagonal.

The resulting algorithm is described in Algorithm 5.

Algorithm 5 A simple fixed-point algorithm to verify the solution to CARE (1.1).

Input A, G, Q;
Compute an approximate solution $\check{X}$ of (1.1) in floating point;
Compute $\tilde{A}$ in floating point as in (4.1);
Compute s and V, for instance $s = -\min\{\Re\lambda : \lambda \text{ is an eigenvalue of } \operatorname{mid}\tilde{\mathbf{A}}\}$ and V as the orthogonal Schur factor of $\operatorname{mid}\tilde{\mathbf{A}}$;
Compute an interval matrix $\mathbf{I}_V$ containing V^{-1};
Compute interval matrices $\mathbf{A}_V, \mathbf{G}_V, \mathbf{Q}_V$ containing A_V, G_V, Q_V;
Compute an interval matrix $\mathbf{I}_s$ containing $(\mathbf{A}_V^* - sI)^{-1}$;
Initialize $\mathbf{Z}_V = -\mathbf{I}_s\mathbf{Q}_V$;
for $k = 1, \dots, k_{max}$ **do**
  Put $\mathbf{Z}_V = \square(0, \mathbf{Z}_V \cdot \texttt{infsup}(0.9, 1.1) + \texttt{realmin} \cdot [-1, 1]$;{$\epsilon-$inflation technique}
  Set $\mathbf{Y} = \mathbf{I}_s(-\mathbf{Q}_V - \mathbf{Z}_V(\mathbf{A}_V + sI - \mathbf{G}_V\mathbf{Z}_V))$;
  if $\mathbf{Y} \subset \operatorname{int}\mathbf{Z}_V$ **then**
    break;
  end if
end for
$\mathbf{X} = \check{X} + \mathbf{I}_V^*\mathbf{Z}_V\mathbf{I}_V$;
Output $\mathbf{X}$.

Theorem 4.1. *Algorithm 5 has a cost of $\mathcal{O}(n^3)$ arithmetic operations.*

Proof. Again, all the required operations are matrix-matrix operations between $n \times n$ matrices. The Schur decomposition requires $\mathcal{O}(n^3)$ operations as well, in practice [11]. Hence the total cost is cubic in n; again, assuming that a moderate number of steps is sufficient. □

Once again, uniqueness is not guaranteed, but it can be deduced *a posteriori* from the stabilizing property of the computed inclusion interval $\mathbf{X}$ (when this holds).

5. Numerical experiments

This section presents numerical experiments to validate the algorithms. We compare four different approaches:

(1) The modified Krawczyk approach described in [13], also described here in Section 3.1. We call this approach *Method H* in the following.
(2) The method described in [24] (using the MATLAB implementation `Mn.m` published by its author). We call this procedure *Method M*.
(3) Algorithm 3, using the Krawczyk-based Algorithm 1 for solving the transformed CARE (3.11). This is a combination of the techniques described in Sections 3.2, 3.3 and 3.4. We call this procedure *Method K* (where K stands for Krawczyk).
(4) Algorithm 3, but using the fixed-point Algorithm 1 for solving the transformed CARE (3.11). This is a combination of the techniques described in Sections 3.3 and 4. We call this procedure *Method F* (where F stands for fixed-point).

The algorithms were implemented in MATLAB 2011a with INTLAB v6, using unit round off $u = 2^{-53} \approx 1.1 \times 10^{-16}$, and run on a computer with an Intel core i5 CPU 2.50GHz and 4GB main memory.

The required stabilizing solutions of CAREs are computed using the method described in [24] (ordered Schur + extra-precise Newton refinement).

We ran these algorithms on all the equations from the benchmark set described in [6], which contains experiments taken from the test suite CAREX [4], run with both default and non-default arguments. The experiment number in Tables 5–5 follows the order used in [6]; see [6, Table 1] and [4] for more details. Note that this set of problems is designed to be extremely challenging for CARE solvers, so it is not surprising that the verification algorithms cannot deal with all of them with perfect accuracy.

In order to assess the quality of the enclosures computed in each experiment we use the normwise relative error nre and the geometric average relative precision garp. The first error measure is defined as

$$\mathrm{nre} := \mathrm{mag}\,\frac{\|\mathrm{rad}\,\mathbf{X}\|_F}{\|\mathbf{X}\|_F},$$

in which F denotes the *Frobenius* norm and mag denotes the magnitude of an interval [27]

$$\mathrm{mag}(\mathbf{x}) := \max\{|x|\colon x \in \mathbf{x}\}.$$

Note that the argument of mag is computed in interval arithmetic and is itself of type interval. This is the simplest possible bound for the (normwise) relative error

$$\frac{\|X_* - \operatorname{mid}\mathbf{X}\|_F}{\|X_*\|_F}$$

obtained by taking $\operatorname{mid}\mathbf{X}$ as an approximation of the solution.

Following the previous work in literature, we also report a componentwise error indicator garp based on the relative precision of an interval, $\operatorname{rp}(\mathbf{X}_{ij})$, defined as

$$\operatorname{rp}(\mathbf{X}_{ij}) := \min(\operatorname{relerr}(\mathbf{X}_{ij}), 1),$$

where relerr is the realtive error of an interval $\mathbf{X}$ defined by

$$\operatorname{relerr}(\mathbf{X}) := \begin{cases} \left|\frac{\operatorname{rad}\mathbf{X}}{\operatorname{mid}\mathbf{X}}\right|, & \text{if} \quad 0 \notin \mathbf{X}, \\ \operatorname{rad}\mathbf{X}, & \text{otherwise.} \end{cases}$$

We define our residual measure as the geometric average of $\operatorname{rp}(\mathbf{X}_{ij})$

$$\operatorname{garp} := \left(\prod_{i,j=1}^{n} \operatorname{rp}(\mathbf{X}_{ij})\right)^{\frac{1}{n^2}}.$$

The quantity $-\log(\operatorname{rp}(\mathbf{X}_{ij}))$ can be interpreted as the number of known correct digits of an *exact* value contained in $\mathbf{X}_{ij}$; so, loosely speaking, $-\log(\operatorname{garp}(\mathbf{X}))$ represents the average number of known correct digits [10].

When the algorithms are successful, we report in Tables 5–5 the number k of required iterations of the outer Krawczyk loop. If the algorithm breaks down or does not converge within the maximum number of steps (which is 15 for Methods H and K and 50 for Method F), we write a star. Method M is not iterative, therefore for it we leave empty the column containing the number of iterations.

The size of the problem (value of n) and the total time (in seconds) taken on our test machine are reported, too, as well as the norm-2 condition number of V (used by Methods H and M) and the same quantity for the eigenvector matrix V_p of the closed-loop matrix $A_P - G_P Y$ used in the other two algorithms.

Remarks are in order on some of the problems.

Experiment 1: This is an example of the phenomenon described in the beginning of Section 4: the closed-loop matrix $A - GX$ associated with the (exact) stabilizing solution is not diagonalizable. The coefficient matrices for this example are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The exact value of the closed loop matrix for the original and transformed equations are respectively

$$A - GX = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad A_p - G_p Y = \begin{bmatrix} -2/3 & 1/3 \\ -1/3 & -4/3 \end{bmatrix},$$

both with a double (defective) eigenvalue in -1. Hence, the computed condition numbers of V and V_p are both large, and the first three algorithms, which are based on diagonalization, fail. On the other hand, the fixed-point algorithm does not encounter any difficulty.

Experiment 17: For this problem, several algorithms report errors on the order of 10^{-300}, which the reader could find surprising. The reason for this is that our solution method returns $\check{X}$ equal to the exact solution matrix X, which is

$$\check{X} = X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since its entries are small integer numbers which are represented exactly in IEEE arithmetic, most of the arithmetic operations are exact and the only operation which enlarges the interval is the ε-inflation technique, in which the interval size is increased by the quantity `realmin` $\approx 2.2 \cdot 10^{-308}$.

Experiments 30 and 31: In Method F for problem 30 and Method K for problem 31, we report termination in a finite number of iterations, but NaN for the error. In these problems, the verification algorithm succeeds for the Riccati equation (3.11), but the resulting interval $\mathbf{Y}$ cannot be converted into a solution interval $\mathbf{X}$ for (1.1) using Lemma 3.5, because the interval matrix $\mathbf{U}_1$ computed as described in Section 3.3 is not invertible. So the method fails to produce a solution enclosure for (1.1).

Another interesting observation is that Method K, when it works, needs only one iteration, i.e., the crucial relation (3.3) is already fulfilled for $k = 1$ in all our examples.

Note that in only two examples with numbers 18 and 24 all methods fail, even though the benchmark [4] consists of many hard problems from different areas.

When they are succesful, Methods H and K are comparable with respect to execution time as well as with respect to the quality of the enclosure. However, there are cases in which Method H is not successful, and this comprises cases with small dimensions (e.g. 2 in Example 10) as well as cases with large dimensions (e.g. 397 in Example 27).

Methods K and M are the most reliable, and fail only on very ill-conditioned examples. Interestingly, the errors obtained by the two approaches differ by

orders of magnitude on several problems, in both directions; there are also examples in which either one fails while the other succeeds. There is no clear winner among the two.

Method F has the largest number of failures. Despite that, it is useful in special cases (such as in Experiment 1) in which the other algorithms have difficulties, particularly when the closed-loop matrix is not diagonalizable.

As one can predict from the theory, in many of the examples the performance of the methods based on diagonalizing the closed-loop matrix is (loosely) related to the condition number of V (or V_p, when it is used). To visualize this relationship, we show in Figures 1 and 2 a scatter plot of the obtained accuracy vs. the value of this condition number in the various examples.

TABLE 1. Comparison between various proposed methods

Experiment number in [4]	Problem property size		Method H		Method M		Method K		Method F	
	cond(V)	cond(V_p)	nre garp	k time	nre garp	k time	nre garp	k time	nre garp	k time
1	2		NaN	*	NaN	-	NaN	*	2.19e-15	2
	1.80e+16	7.75e+07	NaN	*	NaN	*	NaN	*	2.42e-15	1.19e-01
2	2		1.10e-13	1	4.64e-15	-	1.50e-14	1	9.56e-15	2
	1.01e+01	1.15e+00	1.18e-13	7.32e-02	4.97e-15	3.59e-02	1.57e-14	9.23e-02	1.01e-14	1.07e-01
3	4		3.88e-14	1	2.98e-15	-	3.88e-14	1	7.75e-14	4
	9.73e+00	5.10e+00	2.76e-14	9.08e-02	2.11e-15	4.81e-02	5.26e-14	1.31e-01	1.00e-13	1.69e-01
4	8		1.01e-14	1	2.33e-15	-	7.63e-14	1	8.80e-14	8
	1.22e+00	2.17e+00	1.49e-14	7.35e-02	3.42e-15	4.73e-02	1.03e-13	1.11e-01	1.19e-13	1.99e-01
5	9		6.66e-14	1	1.10e-14	-	4.26e-13	1	7.19e-13	9
	7.54e+01	6.52e+01	4.30e-14	7.50e-02	1.05e-14	4.78e-02	7.42e-13	1.11e-01	1.42e-12	4.09e-01
6	30		4.81e-13	2	3.33e-14	-	9.05e-09	1	NaN	*
	1.11e+05	3.48e+03	2.90e-11	1.70e-01	1.83e-12	5.81e-02	1.12e-08	2.05e-01	NaN	*
7	2		2.39e-16	3	2.12e-16	-	5.57e-16	1	5.75e-16	2
	1.62e+00	3.30e+00	5.66e-16	1.05e-01	4.36e-16	3.42e-02	6.45e-16	9.74e-02	8.43e-16	1.08e-01
8	2		9.77e-16	1	1.77e-08	-	3.66e-16	1	8.54e-16	2
	1.00e+00	2.41e+00	1.05e-15	7.05e-02	3.03e-10	4.30e-02	4.82e-16	9.78e-02	1.06e-15	1.09e-01
9	2		6.40e-16	1	1.87e-16	-	3.33e-10	1	3.75e-10	2
	1.22e+00	6.98e+01	2.58e-15	7.04e-02	3.27e-16	4.56e-02	3.36e-10	1.01e-01	3.77e-10	1.52e-01
10	2		NaN	*	1.62e-11	-	2.99e-08	1	1.27e-08	2
	6.80e+01	1.10e+00	NaN	*	1.62e-11	4.32e-02	2.99e-08	9.61e-02	1.27e-08	4.39e-01
11	2		4.95e-16	1	6.28e-16	-	1.22e-15	1	1.46e-15	2
	3.73e+00	1.00e+00	5.32e-16	9.22e-02	6.75e-16	3.63e-02	1.30e-15	1.17e-01	1.56e-15	1.19e-01

TABLE 2. Comparison between various proposed methods

Experiment number in [4]	Problem property: size		Method H		Method M		Method K		Method F	
			nre	k	nre	k	nre	k	nre	k
	cond(V)	cond(V_p)	garp	time	garp	time	garp	time	garp	time
12	2		4.54e-16	1	3.22e-16	-	1.52e-15	1	3.90e-11	2
	1.41e+03	1.00e+00	5.11e-16	8.86e-02	3.73e-16	4.36e-02	2.20e-15	1.17e-01	6.56e-11	1.19e-01
13	2		4.66e-16	1	2.22e-09	-	6.98e-16	1	NaN	*
	2.41e+00	1.00e+00	5.97e-16	7.17e-02	7.19e-11	3.57e-02	1.20e-15	9.84e-02	NaN	*
14	2		1.22e-15	1	2.68e-16	-	9.49e-16	1	4.30e-15	2
	1.00e+00	1.00e+00	1.40e-15	6.73e-02	3.13e-16	3.66e-02	1.09e-15	9.88e-02	5.00e-15	1.09e-01
15	2		3.93e-11	1	3.36e-12	-	3.18e-15	1	NaN	*
	1.00e+00	1.28e+00	3.93e-11	7.06e-02	3.36e-12	3.55e-02	2.62e-15	1.00e-01	NaN	*
16	2		NaN	*	5.87e-10	-	3.78e-15	1	NaN	*
	1.00e+00	1.28e+00	NaN	*	5.87e-10	3.55e-02	2.97e-15	9.43e-02	NaN	*
17	2		9.39e-323	1	2.22e-16	-	5.53e-322	1	6.21e-307	2
	1.00e+00	2.62e+00	1.04e-322	8.65e-02	2.22e-16	3.27e-02	5.53e-322	1.15e-01	6.76e-307	1.49e-01
18	2		NaN	*	NaN	-	NaN	*	NaN	*
	1.00e+00	2.62e+00	NaN	*	NaN	*	NaN	*	NaN	*
19	3		3.40e-15	1	2.72e-16	-	3.41e-15	1	6.78e-15	3
	1.00e+00	1.00e+00	1.15e-14	8.24e-02	8.76e-16	3.86e-02	1.11e-14	9.99e-02	2.21e-14	1.12e-01
20	3		3.04e-11	1	1.03e-05	-	4.63e-15	1	8.31e-15	3
	1.00e+00	1.00e+00	3.57e-11	7.02e-02	1.21e-05	4.56e-02	5.94e-12	1.02e-01	1.07e-11	1.14e-01
21	4		1.18e-14	1	4.76e-15	-	1.17e-13	1	3.29e-13	4
	9.01e+00	3.57e+00	1.59e-14	8.83e-02	6.40e-15	4.13e-02	1.28e-13	1.22e-01	3.73e-13	2.18e-01
22	4		3.25e-12	1	3.51e-06	-	4.32e-12	1	NaN	*
	1.22e+01	5.93e+00	5.82e-13	9.07e-02	2.96e-07	4.71e-02	6.99e-13	1.26e-01	NaN	*

TABLE 3. Comparison between various proposed methods

Experiment number in [4]	Problem property		Method H		Method M		Method K		Method F	
	size		nre	k	nre	k	nre	k	nre	k
	cond(V)	cond(V_p)	garp	time	garp	time	garp	time	garp	time
23	4		4.66e-14	1	1.77e-15	-	3.35e-14	1	1.25e-13	4
	1.43e+01	1.77e+00	4.30e-14	1.02e-01	1.71e-15	3.86e-02	4.51e-14	1.19e-01	1.66e-13	2.16e-01
24	4		NaN	*	NaN	-	NaN	*	NaN	*
	1.73e+00	1.73e+00	NaN	*	NaN	*	NaN	*	NaN	*
25	77		4.13e-12	1	3.66e-13	-	4.72e-11	1	3.09e-10	77
	4.98e+01	1.86e+01	3.54e-11	6.08e-01	3.13e-12	3.60e-01	2.65e-10	1.14e+00	1.59e-09	1.40e+00
26	237		1.19e-10	2	4.35e-12	-	4.39e-09	1	2.43e-08	237
	2.41e+02	8.92e+01	2.23e-09	8.10e+00	8.20e-11	2.61e+00	2.72e-08	8.44e+00	1.37e-07	1.16e+01
27	397		NaN	*	6.71e-12	-	1.31e-08	1	1.00e-07	397
	1.31e+02	4.83e+01	NaN	*	2.26e-10	8.49e+00	8.29e-08	2.77e+01	5.82e-07	4.23e+01
28	8		3.93e-15	1	1.52e-15	-	3.93e-15	1	9.69e-15	8
	1.00e+00	1.00e+00	8.02e-15	6.79e-02	3.10e-15	3.60e-02	8.02e-15	1.03e-01	1.98e-14	1.24e-01
29	64		4.40e-13	1	4.49e-14	-	4.31e-13	1	7.16e-13	64
	1.50e+00	1.50e+00	2.10e-07	1.50e-01	2.14e-08	9.92e-02	2.06e-07	2.68e-01	3.41e-07	2.91e-01
30	21		NaN	*	NaN	-	3.88e-04	1	NaN	21
	2.41e+09	2.77e+00	NaN	*	NaN	*	3.76e-04	1.61e-01	NaN	6.20e-01
31	21		NaN	*	NaN	-	NaN	1	NaN	*
	2.41e+09	2.87e+02	NaN	*	NaN	*	NaN	1.82e-01	NaN	*
32	100		6.57e-12	1	1.14e-12	-	6.57e-12	1	NaN	*
	1.00e+00	1.00e+00	2.27e-11	3.66e-01	3.95e-12	2.63e-01	2.27e-11	7.12e-01	NaN	*
33	60		3.63e-14	1	2.66e-13	-	2.74e-10	1	NaN	*
	1.90e+01	1.55e+01	8.40e-14	3.12e-01	6.09e-13	1.43e-01	4.12e-10	5.11e-01	NaN	*

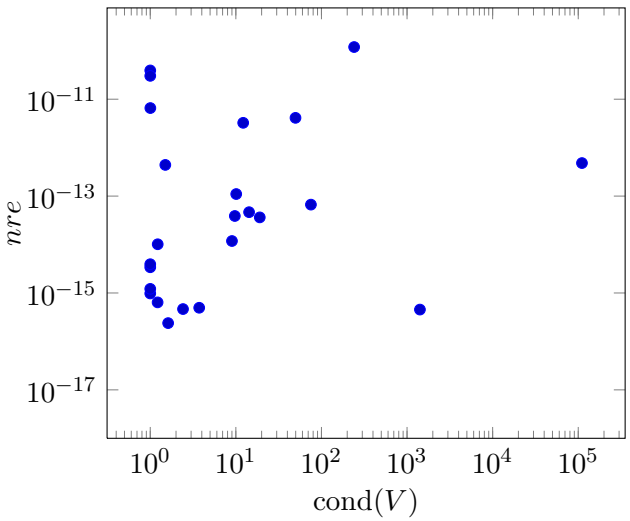

FIGURE 1. nre of Method H vs. cond(V)

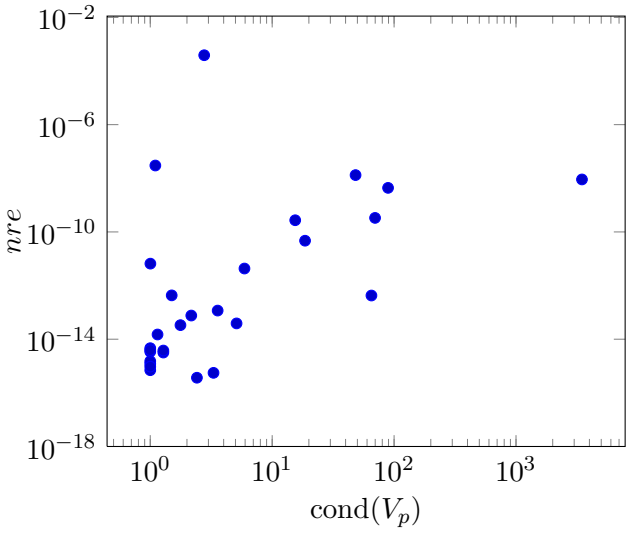

FIGURE 2. nre of Method K vs. cond(V_p)

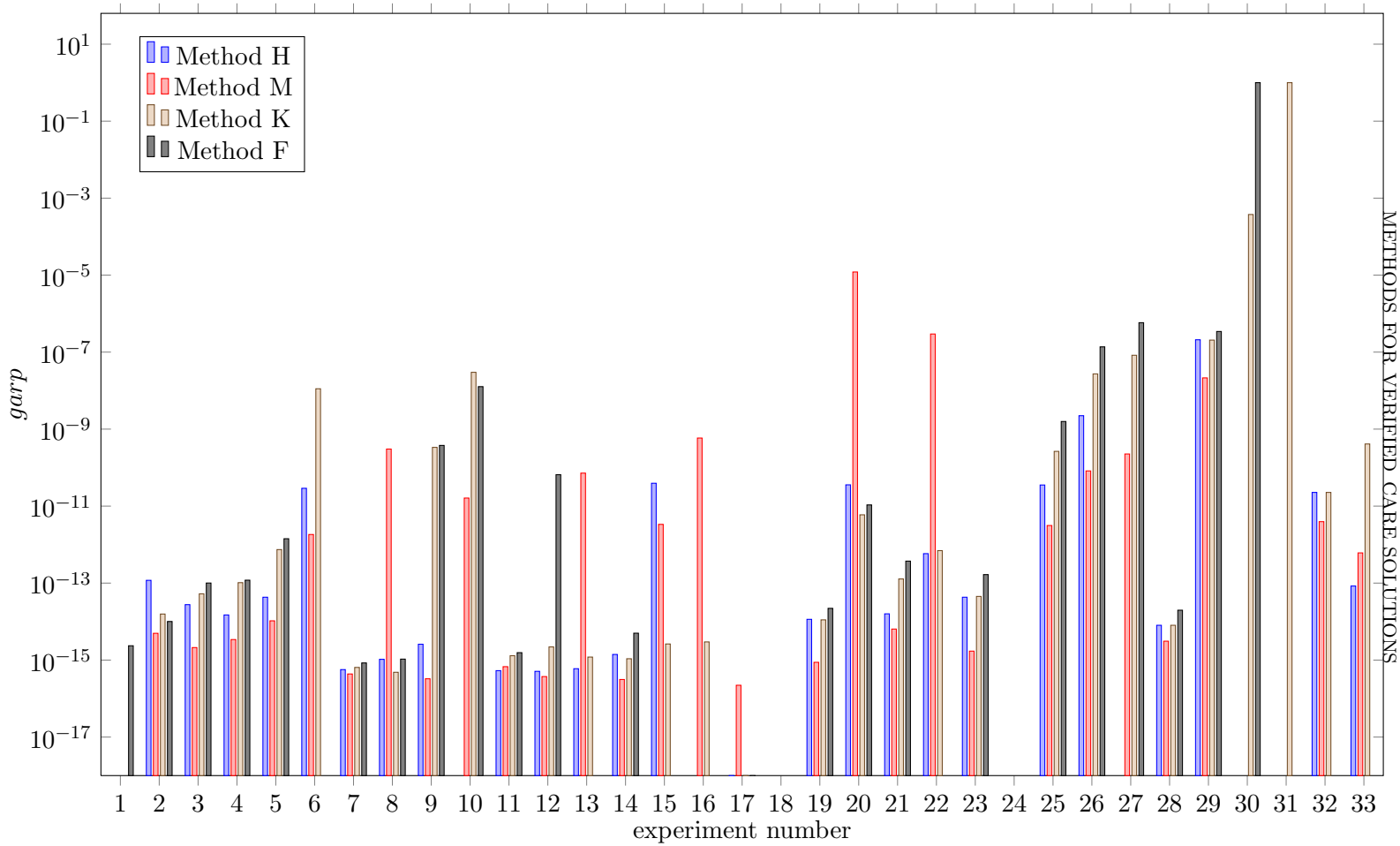

FIGURE 3. garps for each experiment number

We now turn to the verification of the stabilizing property of the computed interval matrix $\mathbf{X}$. For each computed result, we checked its stability using Algorithm 4; the results are reported in Table 5. A plus sign means that the property is verified, a minus sign means failure to verify the property, and a star means that the algorithm had previously failed to compute an inclusion interval. As one can see, there is only a very limited number of cases in which this property cannot be verified. Namely, Experiment 10 (in favor of Method M), Experiment 22 (in favor of Method K), and Examples 30 and 31 (where only Methods K and in one case Method F can verify a solution, but cannot prove its stability).

6. Summary and Outlook

The modified Krawczyk method by Frommer and Hashemi first appeared in [8] is enhanced here with several improvements introduced in Sections 3.2, 3.3, and 3.4. The resulting method is competitive with the one introduced in [24], and returns a smaller solution enclosure in several of the experiments. The new fixed-point method described in Section 4 is a useful addition to the battery of existing verification methods; it is especially useful in the cases in which the closed-loop matrix is not diagonalizable.

There is no single algorithm that beats all the others on all the benchmark problems; hence it is important to have several methods available, each with its strengths and drawbacks. Overall, all but two of the problems in this challenging set of benchmarks could be verified with success.

A number of open problems remain: first of all reducing to zero the number of remaining failures in the methods. Of particular interest would be a method more effective than Algorithm F that does not rely on the closed loop matrix being diagonalizable. Other possible research lines are applying these approaches to discrete-time Riccati equations ($DARE$) or more generally to nonsymmetric algebraic Riccati equations ($NARE$).

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TABLE 4. Results for stabilizing property in all methods

Exp.Num	Method H	Method M	Method K	Method F
1	*	*	*	+
2	+	+	+	+
3	+	+	+	+
4	+	+	+	+
5	+	+	+	+
6	+	+	+	*
7	+	+	+	+
8	+	+	+	+
9	+	+	+	+
10	*	+	-	-
11	+	+	+	+
12	+	+	+	+
13	+	+	+	*
14	+	+	+	+
15	+	+	+	*
16	*	+	+	*
17	+	+	+	+
18	*	*	*	*
19	+	+	+	+
20	+	+	+	+
21	+	+	+	+
22	+	-	+	*
23	+	+	+	+
24	*	*	*	*
25	+	+	+	+
26	+	+	+	+
27	*	+	+	+
28	+	+	+	+
29	+	+	+	+
30	*	*	-	-
31	*	*	-	*
32	+	+	+	*
33	+	+	+	*

References

[1] *IEEE standard for binary floating-point arithmetic*. Institute of Electrical and Electronics Engineers, New York, 1985. Note: Standard 754–1985.

[2] Götz Alefeld and Jürgen Herzberger. *Introduction to interval computations*. Computer Science and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. Translated from the German by Jon Rokne.

[3] R. H. Bartels and G. W. Stewart. Solution of the matrix equation ax + xb = c [f4]. *Commun. ACM*, 15(9):820–826, September 1972. URL: http://doi.acm.org/10.1145/361573.361582, doi:10.1145/361573.361582.

[4] P. Benner, A. Laub, and V. Mehrmann. A collection of benchmark examples for the numerical solution of algebraic Riccati equations I: the continuous-time case. Technical Report SPC 95-22, Forschergruppe 'Scientific Parallel Computing', Fakultät für Mathematik, TU Chemnitz-Zwickau, 1995. Version dated February 28, 1996.

[5] Dario A. Bini, Bruno Iannazzo, and Beatrice Meini. *Numerical solution of algebraic Riccati equations*, volume 9 of *Fundamentals of Algorithms*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2012.

[6] Delin Chu, Xinmin Liu, and Volker Mehrmann. A numerical method for computing the Hamiltonian Schur form. *Numer. Math.*, 105(3):375–412, 2007. URL: http://dx.doi.org/10.1007/s00211-006-0043-0, doi:10.1007/s00211-006-0043-0.

[7] Andreas Frommer. Proving conjectures by use of interval arithmetic. In *Perspectives on enclosure methods (Karlsruhe, 2000)*, pages 1–13. Springer, Vienna, 2001.

[8] Andreas Frommer and Behnam Hashemi. Verified computation of square roots of a matrix. *SIAM J. Matrix Anal. Appl.*, 31(3):1279–1302, 2009. URL: http://dx.doi.org/10.1137/090757058, doi:10.1137/090757058.

[9] Andreas Frommer and Behnam Hashemi. Verified error bounds for solutions of Sylvester matrix equations. *Linear Algebra Appl.*, 436(2):405–420, 2012. URL: http://dx.doi.org/10.1016/j.laa.2010.12.002, doi:10.1016/j.laa.2010.12.002.

[10] Andreas Frommer, Behnam Hashemi, and Thomas Sablik. Computing enclosures for the inverse square root and the sign function of a matrix. *Linear Algebra Appl.*, 456:199–213, 2014. URL: http://dx.doi.org/10.1016/j.laa.2013.11.047, doi:10.1016/j.laa.2013.11.047.

[11] Gene H. Golub and Charles F. Van Loan. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, fourth edition, 2013.

[12] G. I. Hargreaves and Gareth I. Hargreaves. Interval analysis in MATLAB. Technical report, Department of Mathematics, University of Manchester, Manchester, 2002.

[13] Behnam Hashemi. Verified computation of symmetric solutions to continuous-time algebraic Riccati matrix equations. In *SCAN, 15'th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Verified Numerical Computations*, pages 54–56. Russian Academy of Sciences, 2012. With accompanying slides available online. URL: http://conf.nsc.ru/files/conferences/scan2012/139586/Hashemi-scan2012.pdf.

[14] Behnam Hashemi and Mehdi Dehghan. Efficient computation of enclosures for the exact solvents of a quadratic matrix equation. *Electron. J. Linear Algebra*, 20:519–536, 2010.

[15] Behnam Hashemi and Mehdi Dehghan. The interval Lyapunov matrix equation: analytical results and an efficient numerical technique for outer estimation of the united solution set. *Math. Comput. Modelling*, 55(3-4):622–633, 2012. URL: http://dx.doi.org/10.1016/j.mcm.2011.08.036, doi:10.1016/j.mcm.2011.08.036.

[16] Nicholas J. Higham. *Functions of matrices*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. Theory and computation. URL: http://dx.doi.org/10.1137/1.9780898717778, doi:10.1137/1.9780898717778.

[17] Roger A. Horn and Charles R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.

[18] R. B. Kearfott, M. T. Nakao, A. Neumaier, S. M. Rump, S. P. Shary, and P. V. Hentenryck. Standardized notation in interval analysis. In *Proc. XIII Baikal International School-seminar Optimization methods and their applications*, volume 4, pages 106–113, 2005.

[19] Peter Lancaster and Leiba Rodman. *Algebraic Riccati equations*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[20] W. Luther, W. Otten, and H. Traczinski. *Verified Calculation of Solutions of Continuous and Discrete Time Algebraic Riccati Equation*. Number 422 in Schriftenreihe des Fachbereichs Mathematik. Universität Duisburg, Fachbereich Mathematik, 1998.

[21] Wolfram Luther and Werner Otten. Verified calculation of the solution of algebraic Riccati equation. In *Developments in reliable computing (Budapest, 1998)*, pages 105–118. Kluwer Acad. Publ., Dordrecht, 1999.

[22] Volker Mehrmann and Federico Poloni. Doubling algorithms with permuted Lagrangian graph bases. *SIAM J. Matrix Anal. Appl.*, 33(3):780–805, 2012. URL: http://dx.doi.org/10.1137/110850773, doi:10.1137/110850773.

[23] Shinya Miyajima. Fast enclosure for all eigenvalues and invariant subspaces in generalized eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 35(3):1205–1225, 2014. URL: http://dx.doi.org/10.1137/140953150, doi:10.1137/140953150.

[24] Shinya Miyajima. Fast verified computation for solutions of continuous-time algebraic Riccati equations. *Jpn. J. Ind. Appl. Math.*, 32(2):529–544, 2015. URL: http://dx.doi.org/10.1007/s13160-015-0178-4, doi:10.1007/s13160-015-0178-4.

[25] Ramon E. Moore, R. Baker Kearfott, and Michael J. Cloud. *Introduction to interval analysis*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009. URL: http://dx.doi.org/10.1137/1.9780898717716, doi:10.1137/1.9780898717716.

[26] Federico Poloni. PGDoubling – a MATLAB package to solve algebraic Riccati equations and optimal control problems using permuted graph bases, 2012. URL: https://bitbucket.org/fph/pgdoubling.

[27] Siegfried M. Rump. Verification methods: rigorous results using floating-point arithmetic. *Acta Numer.*, 19:287–449, 2010. URL: http://dx.doi.org/10.1017/S096249291000005X, doi:10.1017/S096249291000005X.

[28] S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. URL: http://www.ti3.tuhh.de/rump/.

[29] Eugene L. Wachspress. Iterative solution of the Lyapunov matrix equation. *Appl. Math. Lett.*, 1(1):87–90, 1988. URL: http://dx.doi.org/10.1016/0893-9659(88)90183-8, doi:10.1016/0893-9659(88)90183-8.

[30] K. Yano and M. Koga. Verified numerical computation in lq control problem. *Transactions of the Society of Instrument and Control Engineers*, 45:261–267, 2011.

Department of Mathematics, School of Mathematics and Computer Sciences,, Shahid Bahonar University of Kerman, P. O. Box 76169-14111, Kerman, Iran, Haqiri@math.uk.ac.ir

Dipartimento di Informatica, Università di Pisa,, Largo B. Pontecorvo 3, 56127 Pisa, Italy, fpoloni@di.unipi.it