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MATERIALE DIDATTICO

Abate, "Algebra lineare" ←

Strang "Introduction to linear algebra" ← +Esercizi

Dispense Gaiffi

Video:

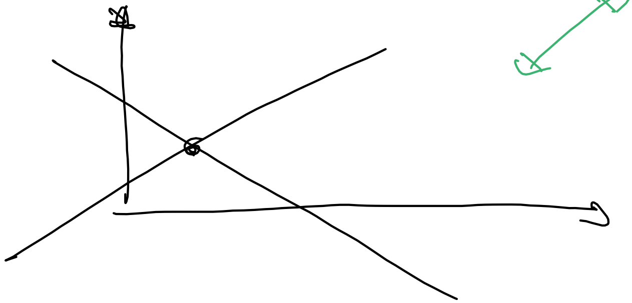
1. Gobbino
2. Strang

Cos'è l'algebra lineare

1. Risolvere sistemi lineari

$$\boxed{\begin{cases} 2x + y = 5 \\ 3x - 5y = 7 \end{cases}}$$

2.



vettori, geometria analitica

3. $x_1, x_2 \dots \quad x_{24}$

$$\underline{a_1x_1 + a_2x_2 + \dots + a_{24}x_{24}}$$

$\begin{cases} \text{10.000 equazioni lineari} \\ \quad\text{vs} \\ \text{50 equazioni di } 2^\circ \text{ grado} \end{cases}$

PageranK

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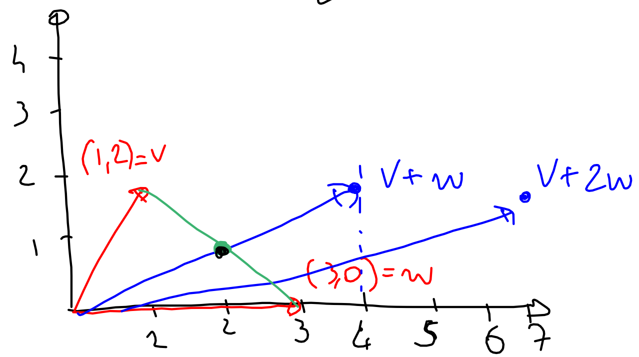
$$\left.\begin{bmatrix} 1 \\ 2 \\ 3 \\ -13 \end{bmatrix}\right\} \text{lunghezza } n \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Somme

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \\ a_4+b_4 \end{bmatrix} \qquad 5 \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 5a_1 \\ 5a_2 \\ 5a_3 \\ 5a_4 \end{bmatrix}$$



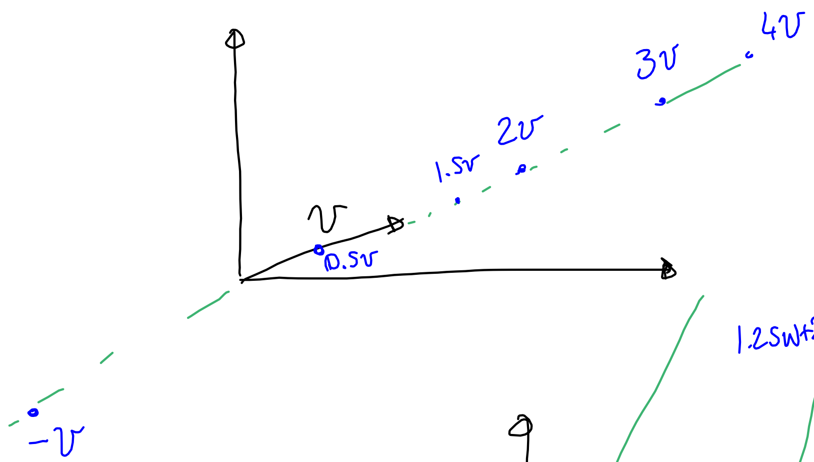
punto medio $= \dfrac{v+w}{2} = \dfrac{(1,2)+(3,0)}{2} = \frac{1}{2}(4,2) = (2,1)$

$v+2w = (1,2) + 2\cdot(3,0) = (7,2)$

Combinazioni lineari

non (0,0)

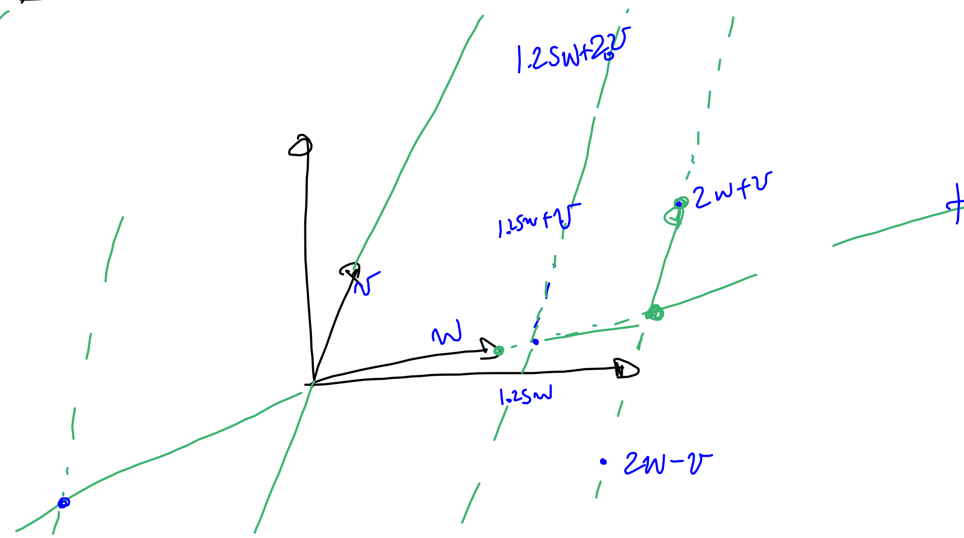
combinazioni lineari di 1 vettore v:
tutta la retta che lo contiene



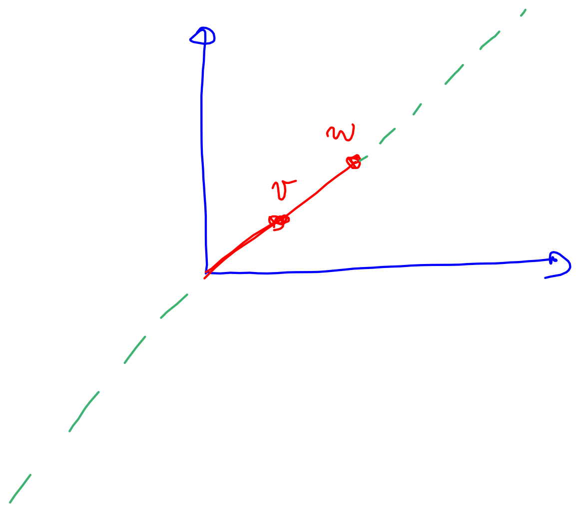
$\cdot\, 2w+5v$

Combinazioni lineari
di 2 vettori:

tutto il piano (di solito)



Perché "di solito"?



$v = (1, 1)$

$w = (2, 2)$

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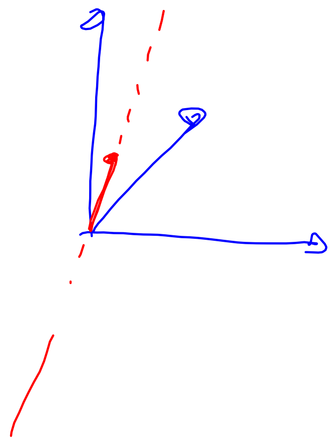
Nello spazio (informalmente):

1 vettore u $c\cdot u$ retta

2 vettori u,v $c\cdot u+d\cdot v$ piano

3. vettori u,v,w $c\cdot u+d\cdot v+e\cdot w$ spazio

(di solito)

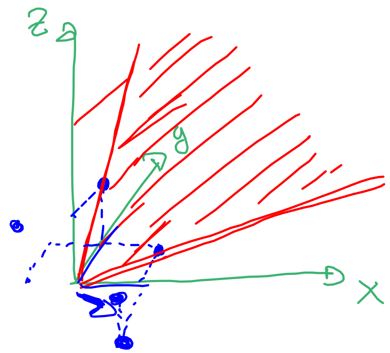


$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

combinazioni lineari: $S = \left\{ cu + dv = \begin{bmatrix} c \\ c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ d \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix} \quad \forall\, c, d \in \mathbb{R} \right\}$

descrizione alternativa: 2^a componente $= 1^a + 3^a$

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + z = 0 \right\}$$



$\{x-y+z=0\}$

Prodotto scalare

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

$$u \cdot v = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$u \cdot v$

$\langle u, v \rangle$ $\quad (u, v)$

dot product

prodotto vettore: non se ne parla (specifico per 3 dimensioni)

$$x-y+z=0 \iff \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

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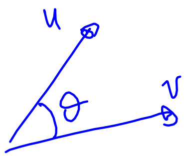
$$v \cdot w = w \cdot v$$

Simmetrico

$$(\underbrace{c}_{\text{numero reale}} \cdot \overbrace{u}^{\text{vettore}} + \underbrace{d}_{\text{num. reale}} \cdot \overbrace{v}^{\text{vettore}}) \cdot w = c(u \cdot w) + d(v \cdot w)$$

distributivo

Norma di un vettore



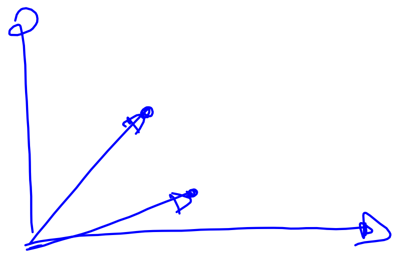
$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|u\| = \sqrt{1^2 + (-1)^2} =$$

$$= \sqrt{2}$$



1) basta farlo per vettori <u>unitari</u>

Perché? Se lo dimostro per u, v di lunghezza 1, per ogni c, d numeri reali

$$(cu)\cdot(dv) = cd\,(u\cdot v)$$

Ogni vettore si può scrivere come reale·(vettore di lunghezza 1)

$$u = \begin{bmatrix} 17 \\ 2 \end{bmatrix} \qquad \|u\| = \sqrt{17^2 + 2^2} = \dots \qquad \frac{1}{\|u\|}\,u$$

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$$u \cdot v = \|u\| \, \|v\| \left(\frac{u}{\|u\|} \cdot \frac{v}{\|v\|} \right)$$

1. $(a u) \cdot v = a (u \cdot v)$ ora prendo $a = \frac{1}{\|u\|}$

$$\left(\frac{1}{\|u\|} u \right) \cdot v = \frac{1}{\|u\|} (u \cdot v)$$

2. $u \cdot (b v) = b (u \cdot v)$ scelgo $b = \frac{1}{\|v\|}$

$$u \cdot \left(\frac{v}{\|v\|} \right) = \frac{1}{\|v\|} (u \cdot v)$$

$$(u \cdot v)^{①} = \|u\| \left(\frac{u}{\|u\|}\right) \cdot v^{②} = \|u\|\,\|v\| \boxed{\left(\frac{u}{\|u\|}\right) \cdot \left(\frac{v}{\|v\|}\right)}$$

resta da dimostrare che questo fa $\cos\theta$

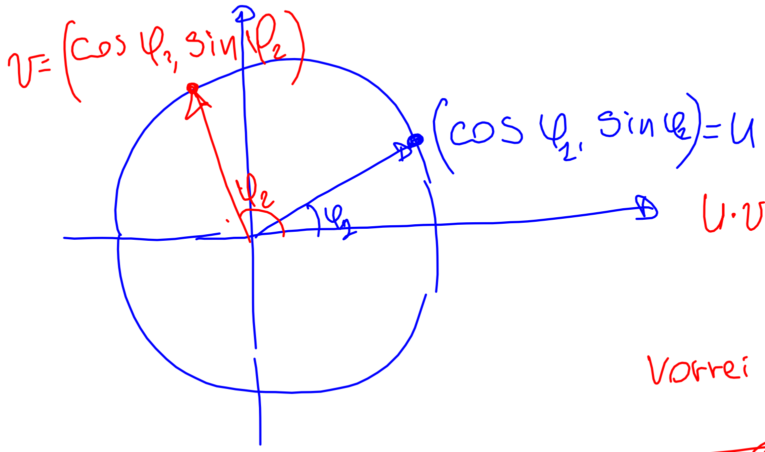
$\frac{u}{\|u\|}$ ha norma 1: perché

$$\left\| \begin{bmatrix} u_1/\|u\| \\ u_2/\|u\| \\ u_3/\|u\| \\ \vdots \\ u_n/\|u\| \end{bmatrix} \right\| = \sqrt{\left(\frac{u_1}{\|u\|}\right)^2 + \left(\frac{u_2}{\|u\|}\right)^2 + \dots + \left(\frac{u_n}{\|u\|}\right)^2}$$

$$= \sqrt{\frac{1}{\|u\|^2}\left(u_1^2 + u_2^2 + \dots + u_n^2\right)} = \sqrt{\frac{1}{\|u\|^2}} \cdot \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$= \frac{1}{\|u\|} \cdot \|u\| = 1$$

Se sono su un piano,



$$u \cdot v = \cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2$$

Vorrei dire che è $\cos(\varphi_2 - \varphi_1)$

$$\left| \frac{u \cdot v}{\|u\| \, \|v\|} \right| = |\cos\theta| \leq 1$$

Quindi $|u \cdot v| \leq \|u\| \|v\|$ Disuguaglianza di Cauchy-Schwarz

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$|u_1 v_1 + u_2 v_2 + \ldots + u_n v_n| \leq \sqrt{u_1^2 + u_2^2 + \ldots u_n^2} \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$$

vera per ogni n-upla di numeri reali

Basta dimostrare che

$$(u_1 v_1 + u_2 v_2)^2 \le (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$ vera

che è equivalente a

$$\cancel{u_1^2 v_1^2} + \cancel{u_2^2 v_2^2} + \underline{2u_1 v_1 u_2 v_2} \le \cancel{u_1^2 v_1^2} + u_1^2 v_2^2 + u_2^2 v_1^2 + \cancel{u_2^2 v_2^2}$$ vera

che è equivalente a

$$0 \le u_1^2 v_2^2 + u_2^2 v_1^2 - 2(u_1 v_2)(v_1 u_2)$$ vera

$$0 \le (u_1 v_2 - u_2 v_1)^2.$$ vera (per $n=2$) $\square$

$n=3$?

$$(u_1v_1+u_2v_2+u_3v_3)^2 \leq (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)$$

$$\cancel{u_1^2v_1^2}+\cancel{u_2^2v_2^2}+\cancel{u_3^2v_3^2}+\underline{2u_1v_1u_2v_2}+2\underline{u_2v_2u_3v_3}+2\underline{u_3v_3u_1v_1} \leq \cancel{9}\ 6 \text{ termini}$$

$$(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2+(u_2v_3-u_3v_2)^2$$

$n=3$ $\square$

$n=4$, $n=5$, ... analoghe

$$2u_iv_iu_jv_j \leq u_i^2u_j^2+u_j^2u_i^2$$

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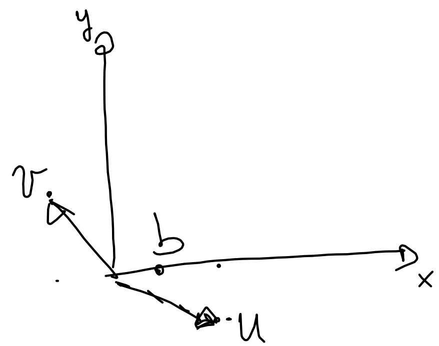

$$u=\begin{bmatrix}2\\-1\end{bmatrix} \quad v=\begin{bmatrix}-1\\2\end{bmatrix}$$

$$b=\begin{bmatrix}1\\0\end{bmatrix}$$

Esiste una combinazione lineare di u, v che dà b?

$$c\begin{bmatrix}2\\-1\end{bmatrix}+d\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}2c-d\\-c+2d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{cases}2c-d=1\\-c+2d=0\end{cases}$$

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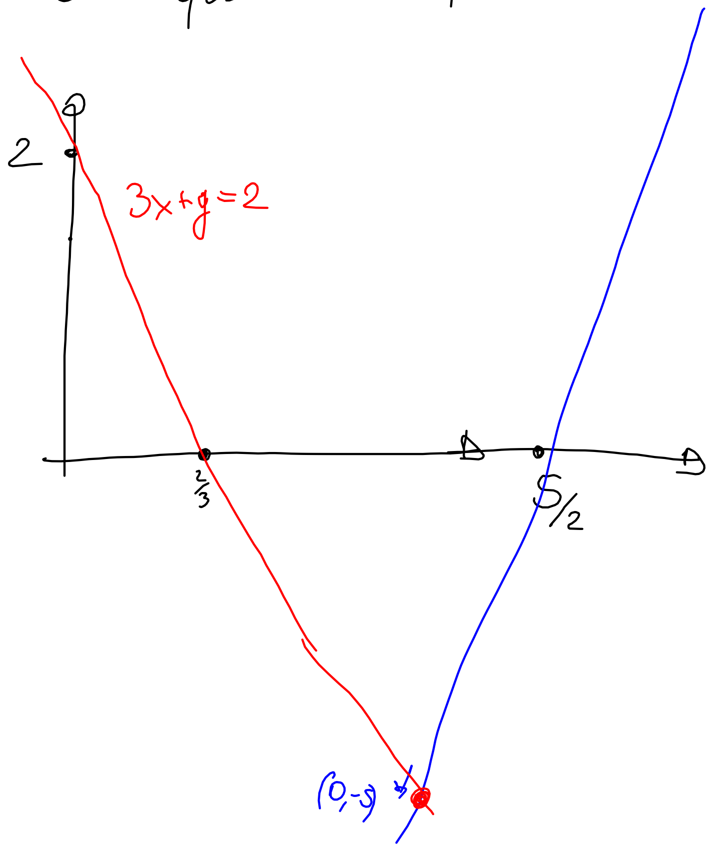
uguaglianza tra vettori $\longleftrightarrow$ sistema di equazioni lineari

$$\begin{cases} \boxed{3x+y}=2 \\ 2x-y=5 \end{cases}$$

$$x\begin{bmatrix}3\\2\end{bmatrix}+y\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$$



$$\begin{cases} 3x+y=2 \\ 2x-y=5 \end{cases}$$

$$\underbrace{\begin{bmatrix} \boxed{3 \quad 1} \\ 2 \quad -1 \end{bmatrix}}_{\text{matrice}} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

prodotto matrice-vettore

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ 2x-y \end{bmatrix}$$

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