Network Analysis with matrices

For us a Network is an undirected, unweighted graph G with N nodes.
Usually represented through a symmetric adjacency matrix $A \in \mathbb{R}^{N \times N}$

Many different centrality measures

- $deg(i) = \sum_{j=1}^{N} a_{ij} = (Ae)_i$ is the degree of node i
- eigenvector centrality $f_i = \frac{1}{\lambda_1} \sum_{j=1}^{N} a_{ij} f_j = \left(\frac{1}{\lambda_1} A f \right)_i$, where λ_1 and f is the Perron-Frobenius eigenpair.

Centrality measures

For any positive integer k, $A^k(i,j)$ counts the number of walks of length k in G that connect node i to node j.

A walk is an ordered list of nodes such that successive nodes in the list are connected. The nodes need not to be distinct.

The length of a walk is the number of edges that form the walk.

Centrality measures

Katz measure

$$k_i = \sum_{j=1}^{N} \sum_{k=1}^{\infty} \alpha^k (A)_{ij}^k = ((I - \alpha A)^{-1} - I)e)_i$$

We can introduce another centrality measure

$$c(i) = (exp(A))_{ii}$$

where the matrix function $exp(A)$ is defined as

$$exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

$c(i)$ accounts for the number of walks of any length from i to i, penalizing long walks respect to shorter ones.

Communicability and Betweenness

Communicability :The idea of counting walks can be extended to the case of a pair of distinct nodes, i and j.

$$C(i,j) = (exp(A))_{ij}$$

Betweenness : How does the overall communicability change when a node is removed?
Let $A - E(r)$ the adjacency matrix of the network with node r removed

$$B(r) = \frac{1}{(N-1)^2 - (N-1)} \sum_{i \neq j, i \neq r, j \neq r} \frac{\exp(A)_{ij} - exp(A - E(r))_{ij}}{(\exp(A))_{ij}}$$

f-centrality

We can extend the concept of centrality/communucability to $c(i) = \sum_{k=1}^{\infty} c_k (A^k)_{ii}$. Adding the coefficient c_0 if the series is convergent for any adjacency matrix A, taking

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, c_k \geq 0$$

we can define

- f-centrality as $c(i) = f(A)_{ii}$
- f-communicability as $C(i,j) = f(A)_{ij}$

f-centrality

We can express A in terms of its spectrum $(\lambda_1 \geq \lambda_2 \leq \cdots \geq \lambda_N$
$A = \sum_{k=1}^{N} \lambda_k x_k x_k^T$ so we have

- f-centrality
$$c(i) = \sum_{k=1}^{N} f(\lambda_k)(x_k(i))^2,$$
- f-communicability
$$C(i,j) = \sum_{k=1}^{N} f(\lambda_k) x_k(i) x_k(j).$$

We can for example take the function

$$r(x) = \left(1 - \frac{x}{N-1}\right)^{-1}$$

In the case of large and sparse networks, $\lambda_k \in [-(N+2), N-2]$, and

$$c(i) = \sum_{k=1}^{N} \frac{N-1}{N-1-\lambda_k} x_k(i)^2,$$

Graph Laplacian and Spectral clustering

Problem : partition nodes into two groups so that we have high intra-connection and low inter-connections

Let $x \in \mathbb{R}^N$ be an indicator vector $x_i = 1/2$ if i belongs to the first cluster, $x_i = -1/2$ if i otherwise.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} (x_i - x_j)^2 a_{ij}$$

counts the number of edges through the cut.
Relax the problem

$$\min_{x \in \mathbb{R}^N : \|x\|_2 = 1 \sum_i x_i = 0} \sum_{j=1}^{N} (x_i - x_j)^2 a_{ij}$$

Let $D = diag(deg(i))$, we have

$$\min_{x\in\mathbb{R}^N:\|x\|_2=1\ \sum_i x_i=0} x^T(D-A)x.$$

The matrix $D - A$ is colled the Graph Laplacian

- $(D - A)e = 0$ so 0 is eigenvalue and the corresponding eigenvector is e
- $D - A$ has nonegative eigenvalues, and the algebric multiplitity of $\mu_1 = 0$ is the number of connected components of the graph
- if the graph is connected $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_N$ with eigenvectors $e = v_1, v_2, \ldots v_N$, the v_2 solves the optimization problem

$$v_2 = \operatorname*{argmin}_{x\in\mathbb{R}^N:\|x\|_2=1\ \sum_i x_i=0} x^T(D-A)x.$$

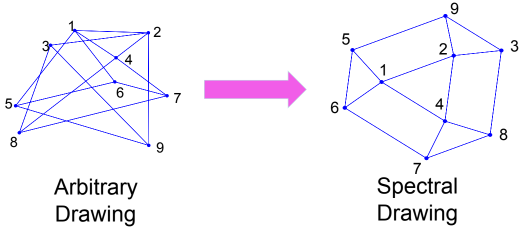
v_2 is called the Fiedler vector of the graph.

Fiedler vector

The Fiedler vector can be the used to

- cluster nodes into two sets, $v_2(i)v_2(j) > 0$, i, j belongs to the same cluster.
- reordering nodes in such a way $i \leq j \implies v_2(i) \leq v_2(j)$
- μ_2 is big iff G has not good clusters
- μ_2 is smal iff G has good clusters

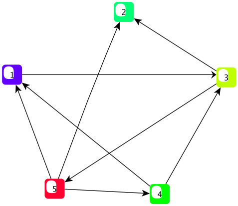
Graph drawing: use spectral coordinates $(v_2(i), v_3(i))$ to draw the graph



Web Graph

The Web is seen as a directed graph:

- Each page is a node
- Each hyperlink is an edge



$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

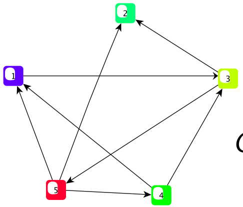
Google's PageRank

- Is a static ranking schema
- At query time relevant pages are retrieved
- The ranking of pages is based on the PageRank of pages which is precomputed
- A page is important if is voted by important pages
- The vote is expressed by a link

PageRank

- A page distribute its importance equally to its neighbours
- The importance of a page is the sum of the importances of pages which points to it

$$\pi_j = \sum_{i \in \mathcal{I}(j)} \frac{\pi_i}{\text{outdeg}(i)}$$



$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}$$

P is row stochastic, $\sum_{j=1}^{N} p_{ij} = 1$.

It is called Random surfer model

The web surfer jumps from page to page following hyperlinks. The probability of jumping to a page depends of the number of links in that page.

Starting with a vector $\pi^{(0)}$, compute

$$\pi_j^{(k)} = \sum_{i \in \mathcal{I}(j)} \pi_i^{(k-1)} p_{ij}, \quad p_{ij} = \frac{1}{\text{outdeg}(i)}$$

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Equivalent to compute the left eigenvector of P corresponding to eigenvalue 1.

PageRank

Two problems:

- Presence of dangling nodes
  - P cannot be stochastic
  - P might not have the eigenvalue 1
- Presence of cycles
  - The random surfer get trapped
  - more than an eigenvalue equal to the spectral radius

Perron-Frobenius Theorem

Let $A \geq 0$ be an irreducible matrix

- there exists an eigenvector equal to the spectral radius of A, with algebraic multiplicity 1
- there exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $A\mathbf{x} = \rho(A)\mathbf{x}$.
- if $A > 0$, then $\rho(A)$ is the unique eigenvalue with maximum modulo.

The same as the ergoodic theorem for Markov chians

PageRank

Presence of dangling nodes

$$\bar{P} = P + \mathbf{d}\mathbf{v}^T$$

$$d_i = \begin{cases} 1 & \text{if page } i \text{ is dangling} \\ 0 & \text{otherwise} \end{cases} \qquad v_i = 1/n;$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \end{bmatrix} \bar{P} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}$$

PageRank

Presence of cycles

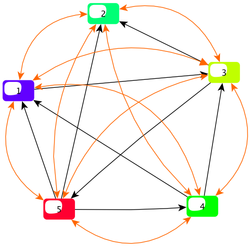
Force irreducibility by adding artificial arcs chosen by the random surfer with “small probability” α.

$$\hat{P} = (1-\alpha)\bar{P} + \alpha \mathbf{e}\mathbf{v}^T,$$

$$\hat{P} = (1-\alpha)\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}.$$

Typical values of α is 0.15.

A toy eample



$$\hat{P} = \begin{bmatrix} 0.05 & 0.05 & 0.8 & 0.05 & 0.05 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.05 & 0.425 & 0.05 & 0.05 & 0.425 \\ 0.425 & 0.05 & 0.425 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.05 & 0.3 & 0.05 \end{bmatrix}$$

Computing the largest left eigenvector of $\hat{P}$ we get

$$\pi^T \approx [0.39, 0.51, 0.59, 0.29, 0.40],$$

which corresponds to the following order of importance of pages

$$[3, 2, 5, 1, 4].$$

PageRank

- P is sparse, $\hat{P}$ is full.
- The vector $y^T = x^T \hat{P}$, for $x \geq 0$, such that $\|x\|_1 = 1$ can be computed as follows

$$\begin{aligned} y^T &= (1-\alpha) x^T P \\ \gamma &= \|x\|_1 - \|y\|_1 = 1 - \|y\|_1, \\ y &= y + \gamma v. \end{aligned}$$

- The eigenvalues of $\bar{P}$ and $\hat{P}$ are related:

$$\lambda_1(\bar{P}) = \lambda_1(\hat{P}) = 1, \quad \lambda_j(\hat{P}) = (1-\alpha)\,\lambda_j(\bar{P}), j > 1.$$

- For the web graph $|\lambda_2(\hat{P})| \leq (1-\alpha)$, $\lambda_2(\hat{P}) = (1-\alpha)$ if the graph has at least two strongly connected components

Generally solved by the power method: rate of convergence $|\lambda_2|/|\lambda_1|$.