$x_k = \frac{(k-1)\ell}{n}$

$x_{k+1} - x_k = k\frac{\ell}{n} - (k-1)\frac{\ell}{n} = \frac{\ell}{n}$

$$\sum_{k=1}^{n} x_{k+1}^2 \cdot \frac{\ell}{n} = \sum_{k=1}^{n} \left(\frac{k\ell}{n}\right)^2 \frac{\ell}{n} = \frac{\ell^3}{n^3} \sum_{k=1}^{n} k^2 =$$

$$= \frac{\ell^3}{n^{\cancel{3}2}} \frac{\cancel{n}(n+1)(2n+1)}{6} = f_s(n,\ell)$$

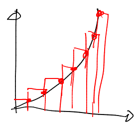
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

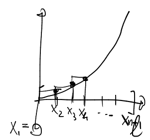
$$\lim_{n\to\infty} f_s(n,\ell) = \frac{\ell^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{\ell^3}{3}$$

$$f_i(n,\ell) = \frac{\ell^3}{n^3}\sum_{k=1}^{n}(k-1)^2 = \frac{\ell^3}{n^3}\sum_{j=0}^{n-1} j^2 =$$

$j = k-1$

$$= \frac{\ell^3}{n^3}\frac{(n-1)(n)(2n-1)}{6} \to \frac{\ell^3}{3}$$





$f(x) = x^2$

$x_k = \frac{(k-1)\ell}{n}$

$$\text{R.S.S.} = \sum_{k=1}^{n}(x_{k+1}-x_k)\,x_{k+1}^2 = f_s(n,\ell)$$

$n = k \cdot 10 \cdot \infty$

$$[f_s(n,\ell) - f_i(n,\ell)] \cdot n \approx 8$$

$$f_s(n,\ell) - f_i(n,\ell) = O\left(\frac{1}{n}\right)$$

$$(f_s(n,\ell) - f_i(n,\ell))\,n \to \ell^3$$

"educated guess"

$1+2+3+\dots+n^2 -$

$0+1+2+\dots(n-1)^2$

$$f_s(n,\ell) - f_i(n,\ell) = \frac{\ell^3}{n^3}\left(\sum_{k=1}^{n} k^2 - \sum_{j=0}^{n-1} j^2\right) =$$

$$= \frac{\ell^3}{n^{\cancel{3}}} \cdot \cancel{n^2} = \frac{\ell^3}{n}$$

$$g(n,\ell) = (f_s(n,\ell) - f_i(n,\ell))\,n = \frac{\ell^3}{n} \cdot n = \ell^3$$