Interval arithmetic methods to verify the stabilizing solution of an algebraic Riccati equation

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Overview

**Goal:** compute a set $\mathbf{X}$ which contains (for sure, not “up to small computational errors”) the stabilizing solution $X_s$ of

$$0 = F(X) = A^\top X + XA + Q - XGX.$$

Do not use more than $O(n^3)$ flops.

**Plan**

- Convince you that interval arithmetic is a good idea.
- Show you what people did to verify Riccati equations.
- Show you the improvements we introduced.
- Competitors, experiments, and other ideas.
Basic idea if $a \in [1, 2]$ and $b \in [3, 4]$, then $a + b \in [4, 6]$ and $ab \in [3, 8]$. Store $(\text{min, max})$ (or $(\text{center, radius})$) and operate on them. $\mathbb{IR}$, $\mathbb{IC}$.

With IEEE arithmetic + rounding in the correct direction, the inclusions work irrespective of machine errors.

Machine numbers can be embedded in $\mathbb{IR}$ as radius-0 intervals.
Wrapping effect

\[
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad v = \begin{bmatrix} [1, 2] \\ [3, 4] \end{bmatrix}
\]

Image of \( v \): blue square. Interval result: red+blue. This happens even though \( \kappa(A) = 1! \)
Verify, don’t solve

The first rule of interval arithmetic

You don’t solve your problem with interval arithmetic.

Things like back-substitution would create huge intervals.
E.g., solving $AX + XB = C$ with Bartels-Stewart: hopeless.

Instead:

$g(x) \subset x$ implies $x = g(x)$ for some $x \in x$

- compute (with usual methods) an approximate solution $\tilde{x}$.
- reformulate as $x = g(x)$, e.g., $x = x - Rf(x)$.
- choose an interval $x \ni \tilde{x}$, e.g., $\tilde{x} - [0.9, 1.1]f(\tilde{x})$
- check (hopefully) $g(x) \subset x$.
- if not, enlarge $x$, e.g., $x \leftarrow [0.9, 1.1]g(x)$ and retry.

Details omitted; e.g.: need care with computing $x - Rf(x)$.
The Krawczyk method

Ingredients:

- approximate solution $\tilde{x}$.
- slope $S_x$: set such that there is $S \in S_x$ satisfying

  $$f(x) - f(\tilde{x}) = S(x - \tilde{x}) \quad \text{for all } x \in x.$$  

  Often related to an interval evaluation of $f'(x)$.
- preconditioner $R$: approximate inverse of some matrix in $S_x$.

Theorem [Krawczyk '69, Rump '83]

If, for some interval $\delta$,

$$\text{int}(\delta) \supseteq -Rf(\tilde{x}) + (I - RS_{\tilde{x}+\delta})\delta,$$

then $\tilde{x} + \delta$ contains a solution of $f(x) = 0$.

If (*) holds replacing $\tilde{x}$ with every $y \in x$, then the solution is unique.
Verifying Riccati equations

\[ F(X) = A^T X + XA + Q - XGX \]

**O(n^3) algorithm:** [Hashemi ’12]

- \( \tilde{X} \) from your favorite method.
- **\( S_x: \)** \( F'(X) = (A - GX)^T \otimes I + I \otimes (A - GX)^T \) works.
- **\( R: \)** can’t use Bartels-Stewart. Instead: explicit eigendecomposition \( (A - GX) \approx VDV^{-1} \) and

\[
R = (V^{-T} \otimes V^{-T})(D^T \otimes I + I \otimes D^T)^{-1}(V^T \otimes V^T)
\]

Additional manipulations: \( (I - RS_x) = (V^{-T} \otimes V^{-T})(\cdots)(V^T \otimes V^T) \)

Again, many details omitted; for instance, dealing properly with \( W \approx V^{-1} \).
Improving Hashemi’s method

Our goal: construct an enclosure $\mathbf{X}$ for the stabilizing solution $\mathbf{X}_s$.

Plan:
- Compute an enclosure $\mathbf{X}$ starting from $\tilde{\mathbf{X}} \approx \mathbf{X}_s$.
- Verify that each matrix in $\mathbf{A} - \mathbf{G}\mathbf{X}$ is stabilizing.
- Uniqueness follows from classical Riccati theory. [Brockett, ’70]

Letting go of uniqueness allows some improvements:
1. Tighter slope $S_{\mathbf{X}}$.
2. Defer the change of basis as in [Frommer Hashemi ’09].
3. Verify a different equation using tricks from [Mehrmann P. ’12].
**Improvements**

1. **Tighter slope $S_X$**
   We can use $S_x = (A - GX)^\top \otimes I + I \otimes (A - G\tilde{X})^\top$.

2. **Defer the change of basis**
   Find $Y$ that encloses a solution of $\hat{F}(Y) = V^\top f(V^{-T} Y V^{-1}) V$:
   Easier, because $\hat{F}'$ is diagonal.

   Then, compute $X = V^{-T} Y V^{-1}$.
   Even if $Y \in Y$ unique solution, other solutions may end up in $X$ due to wrapping effects.

   [Frommer Hashemi '09] introduced this trick for $\sqrtm$. 
Improvements

3 Verify a different equation

\[ \text{CARE} \iff \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (A - GX) \]

[Mehrmann P. ’12]: one can find a basis for \( \text{im} \begin{bmatrix} I \\ X \end{bmatrix} \) with an identity in different position (i.e., \( \text{im} \begin{bmatrix} I \\ X \end{bmatrix} = \text{im} \begin{bmatrix} I \\ Y \end{bmatrix} \), \( II \) permutation matrix) so that \( |Y|_{ij} \leq \sqrt{2} \).

As above, we can verify a Riccati equation for \( Y \) rather than one for \( X \).

Smaller / more balanced entries \( \implies \) easier verification.
Verify a different equation

Algorithm

- Compute approximate CARE solution $\tilde{X}$
- Compute $\Pi$ so that $\text{im} \left[ \frac{I}{\tilde{X}} \right] = \text{im} \Pi \left[ \frac{I}{\tilde{Y}} \right]$, with $\tilde{Y}$ bounded.
- Form the CARE associated with $\Pi^{-1} \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \Pi$ instead of
  $\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}$.
- Compute an inclusion $\bar{Y} \supseteq Y_s$ of its stable solution.
- $X = U_2 U_1^{-1}$, where $\Pi \begin{bmatrix} I \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$. Other solutions may enter $X$. 
Summing up

- Start from an approximate stabilizing solution $\tilde{X}$
- Use the above methods to construct $X \ni \tilde{X}$ containing a solution
- If all the matrices in $X$ are stabilizing, bingo!

Alternative approach (main competitor): [Miyajima '15].
Mix between the above methods and explicit normwise bounds. Idea:
- Newton-like iteration $X = g(X), g(X) = X - (F'_{\tilde{X}})^{-1}(F(X))$.
- Formula for $F'_{\tilde{X}}$ using an eigendecomposition of $A - G\tilde{X}$, as earlier.
- Expand $g(X)$, where $X = (\tilde{X} - \eta R, \tilde{X} + \eta R)$ (for a specific choice of $R$), as a function of $\eta$.
- Using inequalities, determine $\eta$ such $X \supseteq g(X)$ (if possible).
- Compute $\eta$ using interval arithmetic and rounding.
- Uniqueness and stabilizing-ness verified \textit{a posteriori}.
Diagonalizability

Verification methods tested on the benchmarks in CAREX [Benner et al '95]

OK on many of them, but we are still not satisfied:

**CAREX Example 1** [Benner et al '95]

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad X_s = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]

This example can be used for a first verification of any solver for [CAREs] since the solution may be computed by hand.

\[A - GX_s\] is not diagonalizable \(\implies\) all methods fail on this ‘warm-up’ example.
Non-diagonalizable problems

New algorithm: not as effective as the others, but it works in $O(n^3)$ even if $A - GX_s$ is (almost) not diagonalizable.

Idea

- Rewrite as a CARE in $\Delta$, where $X = \tilde{X} + \Delta$:
  \[
  \hat{A}^{\ast} \Delta + \Delta \hat{A} + \hat{Q} - \Delta G \Delta = 0.
  \]
- Mimic ADI: fixed-point eqn $\Delta = (\hat{A} - sl)^{-\top} (\Delta G \Delta - \hat{Q} - \Delta (A + sl))$.
- Are there parameters that we can tune? Choice of $s$, and then change of basis:
  \[
  \Delta_V = V^{\ast} \Delta V, \quad A_V = V^{-1} \tilde{A} V, \quad Q_V = V^{\ast} \tilde{Q} V, \quad G_V = V^{-1} G V^{-\ast}.
  \]
  No need to diagonalize this time.
In practice, we choose $V = \text{orthogonal Schur factor of } \hat{A}$, $s = -\lambda_{\text{max}}(\hat{A})$.
Performance profile on CAREX suite in [Chu et al '07]

Top left = better.
(Norm-2) width of found interval

![Graph showing the comparison between Miyajima and the new method.](image-url)
CPU time on CAREX 15

Lower = better. New = only method to reach \( n = 1000 \).
Conclusions

- Technical improvements and ideas from Riccati theory take Krawczyk-based method to state-of-the-art level.
- No method always better than the others, so it is useful to have more choice.
- In almost all cases, the first solution guess $\tilde{x} - [0.9, 1.1]f(\tilde{x})$ already works — so there is still room to optimize.
- Up next: transfer some of these improvements to Miyajima’s method.
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[Thanks for your attention!]