A projection method for the solution of large-scale Lur’e equations

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Control problems and even matrix pencils

Several problems in control theory (model reduction, positive real lemma) naturally expressed as deflating subspace problems for

Even matrix pencils

\[
\begin{bmatrix}
0 & A & B \\
A^* & Q & S \\
B^* & S^* & R
\end{bmatrix}
- s
\begin{bmatrix}
0 & I & 0 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\(A - sE\) is even, i.e., \(A = A^*, E = -E^*\)

\(n \gg m\), \(A\) large and sparse, \(Q\) low rank

We are looking for the maximal semi-stable \(E\)-neutral deflating subspace, i.e.,

\[
AU = V \hat{A} \quad E U = V \hat{E} \quad U, V \in \mathbb{C}^{2n+m,k} \quad U^* E U = 0
\]
What if $R$ is singular?

The singular $R$ case has been treated stepmotherly \text{(T. Reis)}

- the Riccati equation cannot be formed
- numerical problems: nontrivial Jordan blocks at infinity and/or singular pencil
- in engineering practice, often solved by perturbing+inverting $R$

ARE must be replaced by a system

Lur’e equations

\[
A^T X + XA + Q = Y^T Y
\]
\[
XB + S = Y^T Z
\]
\[
R = Z^T Z
\]

(only $X$ needed in practice)
Lur’e equations and deflating subspaces

Deflating subspace formulation

\[
\begin{bmatrix}
0 & -sl + A & B \\
sl + A^* & Q & S \\
B^* & S^* & R
\end{bmatrix}
\begin{bmatrix}
X & 0 \\
I_n & 0 \\
0 & I_m
\end{bmatrix}
= 
\begin{bmatrix}
I_n & 0 \\
-X & Y^* \\
0 & Z^*
\end{bmatrix}
\begin{bmatrix}
-sl + A & B \\
Y & Z
\end{bmatrix}
\]

\( \ker \mathcal{E} = \begin{bmatrix}
0 \\
0 \\
l_m
\end{bmatrix} \) “obvious” deflating subspace \((\lambda = \infty)\).

Partial subspace

\[
\begin{bmatrix}
V_1 & 0 \\
V_2 & 0 \\
0 & I
\end{bmatrix}
\subseteq U_X \iff \text{Partial solution: } XV_2 = V_1, \quad X = V_1 V_2^+ + \cdots
Even Kronecker canonical form

[Thompson, ’76 & ’91], a powerful tool to analyze Lur’e equations theoretically [Reis, ’11]

Canonical form under transformations of the kind $M^TAM$, $M^TEM$
(for any $M$ nonsingular)

Plays well with
- deflating subspaces $(A - sE)U = V(\hat{A} - s\hat{E})$
- $E$-neutrality $U^TEU = 0$ (and similar relations)

Even Kronecker canonical form [Thompson, ’76 & ’91]

Every even matrix pencil (i.e., $A = A^*$, $E = -E^*$) can be reduced to a
direct sum of the following block types...
Even Kronecker canonical form

\[
\begin{bmatrix}
\lambda - s & 1 \\
\lambda - s & 1 \\
\bar{\lambda} + s & \lambda + s \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

paired eigenvalues \((\lambda, -\bar{\lambda})\)

\[
\begin{bmatrix}
i\mu - s & 1 \\
i\mu - s & 1 \\
i\mu - s & 1 \\
i\mu - s & 1 \\
i\mu - s & 1 \\
\end{bmatrix}
\]

imaginary eigenvalues \(i\mu\)

\[
\begin{bmatrix}
s & 1 \\
s & 1 \\
s & 1 \\
s & 1 \\
s & 1 \\
\end{bmatrix}
\]

eigenvalues at \(\infty\)

\[
\begin{bmatrix}
s & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

singular blocks
The bad guys

$$\begin{bmatrix}
  s & 1 \\
  s & 1 \\
  s & 1 \\
  1 & 1 \\
\end{bmatrix}$$

$$\begin{bmatrix}
  s & 1 \\
  s & 1 \\
  s & 1 \\
  s & 1 \\
\end{bmatrix}$$

eigenvalues at $\infty$
singular blocks

Singular $R \Leftrightarrow$ nontrivial blocks of one of these two kinds.

**Theorem**

For all solutions $X$, $U_X$ contains the first $\frac{\ell-1}{2}$ vectors of each of these Kronecker chains ($\ell=\text{length}$)
Wong sequences

Pencil generalization of the procedure used to compute Jordan chains/bases [Wong KT, '74] [Berger, Ilchmann, Trenn, '10]

**Wong sequence (for \( \lambda = \infty \))**

\[
\mathcal{W}_0 = \{0\}, \quad \mathcal{W}_{k+1} = \mathcal{E}^{-1}(A\mathcal{W}_k)
\]

(The \( \mathcal{W}_i \) are subspaces, and \( \mathcal{E}^{-1} = \text{preimage} \))

Switch to even Kronecker form, everything here transforms well
Wong sequences of Kronecker blocks

\[ \mathcal{W}_3 \mathcal{W}_2 \mathcal{W}_1 \]

\[ \begin{bmatrix}
  s & 1 \\
  s & 1 \\
  s & 1 \\
  1 & 1
\end{bmatrix} \]

\[ \mathcal{W}_1 = \text{span}\{e_n\} \]
\[ \mathcal{W}_2 = \text{span}\{e_{n-1}, e_n\} \]

\( \vdots \)

Problem: how to force them to stop at half the size of each block?

Idea: that’s exactly where they stop being \( \mathcal{E} \)-neutral!
\(\mathcal{E}\)-neutral Wong sequences

Wong sequence (for \(\lambda = \infty\))

\[
\begin{align*}
\mathcal{V}_0 &= \{0\}, \\
\mathcal{Z}_k &= \mathcal{E}^{-1}(A\mathcal{V}_k), \\
\mathcal{V}_{k+1} &= \mathcal{V}_k + \mathcal{Z}_k \cap \mathcal{Z}_k^{\mathcal{E}\perp}
\end{align*}
\]

Theorem

\(\mathcal{E}\)-neutral Wong sequences are increasing (\(\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots\)) and stabilize to the space spanned by the first \(\frac{\ell+1}{2}\) vectors of each infinite (and singular) chain.

\(\mathcal{V}_\infty\) gives a partial solution: \(X\mathcal{V}_2 = \mathcal{V}_1\), \(X = \mathcal{V}_1\mathcal{V}_2^+ + \cdots\) for some \(\mathcal{V}_1, \mathcal{V}_2\).

Question How to compute the remaining part?
Projected Lur’e equations

We multiply everything in the Lur’e equations by $\Pi = I - V_2 V_2^+$, and get

**Theorem**

$\tilde{X} = \Pi^* X \Pi$ satisfies projected Lur’e equations with

\[
\begin{align*}
\tilde{A} &= \Pi A \Pi, \\
\tilde{Q} &= \Pi^* Q \Pi, \\
\tilde{B} &= [\Pi AV_2 \quad \Pi B], \\
\tilde{S} &= \begin{bmatrix} \Pi^* A^* V_1 + \Pi^* QV_2 & \Pi^* S \end{bmatrix}, \\
\tilde{R} &= \begin{bmatrix} V_2^* A^* V_1 + V_1^* AV_2 + V_2^* QV_2 & V_1^* B + V_2^* S \\
B^* V_1 + S^* V_2 & R \end{bmatrix}
\end{align*}
\]
Projected Lur'e equations

“Projection” \(\iff\) zeroing out the critical subspace at infinity

In the right basis,

\[
P \begin{bmatrix}
0 & \tilde{A} - sl & \tilde{B} \\
\tilde{A}^* + sl & \tilde{Q} & \tilde{S} \\
\tilde{B}^* & \tilde{S}^* & \tilde{R}
\end{bmatrix} P^T \cong \begin{bmatrix}
0 & A_1 - sl & B_1 & 0 \\
A_1^* + sl & Q_1 & S_1 & 0 \\
B_1^* & S_1^* & R_1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\(R_1\) nonsingular, so we can turn this into a projected Riccati equation

\[
\begin{bmatrix}
A_{11}^* & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
X_{11} & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
X_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{11} & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
Q_{11} & 0 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
X_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
G_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
X_{11} & 0 \\
0 & 0
\end{bmatrix}
\]

We solve this ARE with Newton-ADI (Lyapack, [Benner, Li, Penzl, '08]).

Problem \(A_{11}\) is dense: we must use \(\tilde{A} = \Pi A \Pi = (I - V_2 V_2^+) A (I - V_2 V_2^+)\)
to preserve sparsity
What happens in ADI

ADI: lots of singular equations with $\Pi A \Pi$:

$$
\begin{bmatrix}
A_R - zI & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
0
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix}
$$

In fact, if we work with $\Pi A \Pi - zI$ we regularize them for free:

$$
\begin{bmatrix}
A_R - zI & 0 \\
0 & -zI
\end{bmatrix}
\begin{bmatrix}
x \\
0
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix}
$$

Further trick: rewrite $(I - V_2 V_2^+) A (I - V_2 V_2^+) x = b$ as extended system

$$
\begin{bmatrix}
A & V_2 & \Pi A V_2 \\
V_2^+ A & I & 0 \\
V_2^+ & 0 & I
\end{bmatrix}
\begin{bmatrix}
x \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
b \\
0 \\
0
\end{bmatrix}
$$

Preserves sparsity, now we can use sparse LU
To sum up

Algorithm

1. Compute $\mathcal{V}_\infty$ “critical subspace” using $\mathcal{E}$-neutral Wong sequences
2. Compute coefficients $\tilde{B}, \tilde{R}, \tilde{S}$ of the projected equation, and sparse representations of $\tilde{A} = \Pi A \Pi$, $\tilde{Q} = \Pi^* Q \Pi$
3. Use Newton-ADI to solve the projected Riccati equation for $\tilde{X}$. Use extended matrix approach for solvers.
4. Assemble solution $X = V_1 V_2^+ + \tilde{X}$

F. Poloni, T. Reis

*On combining deflation and iteration to low-rank approximate solution of Lur’e equations*

Example I

Lur’ë equations from positive real lemma
Demo system demo-r1 in Lyapack (heat equation on the square)

<table>
<thead>
<tr>
<th></th>
<th>demo-r1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>2500</td>
</tr>
<tr>
<td>$m$</td>
<td>1</td>
</tr>
<tr>
<td>rank decisions accuracy</td>
<td>$1.6 \times 10^{-16}$</td>
</tr>
<tr>
<td>infinite chains</td>
<td>$1 \times$ length 3</td>
</tr>
<tr>
<td>singular chains</td>
<td>0</td>
</tr>
<tr>
<td>rank of $X^{(1)}$</td>
<td>24</td>
</tr>
<tr>
<td>rank of $X - X^{(1)}$</td>
<td>23</td>
</tr>
<tr>
<td>no. of Newton steps needed</td>
<td>4</td>
</tr>
<tr>
<td>avg. ADI itns per Newton step</td>
<td>37.25</td>
</tr>
<tr>
<td>relative residual</td>
<td>$2.6 \times 10^{-15}$</td>
</tr>
<tr>
<td>deviation from stability</td>
<td>$-1.8 \times 10^{-15}$</td>
</tr>
<tr>
<td>CPU time</td>
<td>17 s</td>
</tr>
</tbody>
</table>
Example II

Lur’e equations from positive real lemma
Demo system demo-r3 in Lyapack (rail profile)

<table>
<thead>
<tr>
<th></th>
<th>demo-r3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>821</td>
</tr>
<tr>
<td>$m$</td>
<td>6</td>
</tr>
<tr>
<td>rank decisions</td>
<td>$6.5 \times 10^{-16}$</td>
</tr>
<tr>
<td>infinite chains</td>
<td>$6 \times$ length 3</td>
</tr>
<tr>
<td>singular chains</td>
<td>0</td>
</tr>
<tr>
<td>rank of $X^{(1)}$</td>
<td>138</td>
</tr>
<tr>
<td>rank of $X - X^{(1)}$</td>
<td>130</td>
</tr>
<tr>
<td>no. of Newton steps</td>
<td>7</td>
</tr>
<tr>
<td>avg. ADI itns per Newton step</td>
<td>36.857</td>
</tr>
<tr>
<td>relative residual</td>
<td>$5.5 \times 10^{-15}$</td>
</tr>
<tr>
<td>deviation from stability</td>
<td>$-1.3 \times 10^{-08}$</td>
</tr>
<tr>
<td>CPU time</td>
<td>65 s</td>
</tr>
</tbody>
</table>
Example II
Lur’e equations from positive real lemma
Demo system demo-r3 in Lyapack (rail profile)

\begin{align*}
\text{rank of } X - X^{(1)} &= 130 \\
\text{no. of Newton steps needed} &= 7 \\
\text{avg. ADI itns per Newton step} &= 36.857 \\
\text{relative residual} &= 5.5 \times 10^{-15} \\
\text{deviation from stability} &= -1.3 \times 10^{-08} \\
\text{CPU time} &= 65 \text{s}
\end{align*}

Thanks for your attention! Questions?