A projection method for the solution of large-scale Lur'e equations

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Control problems and even matrix pencils

Several problems in control theory (model reduction, positive real lemma) naturally expressed as deflating subspace problems for

Even matrix pencils

$$\mathcal{A} - s\mathcal{E} = \begin{bmatrix} 0 & A & B \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} - s \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{A}, \mathcal{E} \in \mathbb{R}^{n+n+m,n+n+m}$$

$\mathcal{A} - s\mathcal{E}$ is even, i.e., $\mathcal{A} = \mathcal{A}^*$, $\mathcal{E} = -\mathcal{E}^*$

$n \gg m$, A large and sparse, Q low rank

We are looking for the maximal semi-stable $\mathcal{E}$-neutral deflating subspace, i.e.,

$$\mathcal{A}U = V\widehat{\mathcal{A}} \qquad \mathcal{E}U = V\widehat{\mathcal{E}} \qquad U, V \in \mathbb{C}^{2n+m,k} \qquad U^*\mathcal{E}U = 0$$

What if R is singular?

The singular R case has been treated stepmotherly *(T. Reis)*

- the Riccati equation cannot be formed
- numerical problems: nontrivial Jordan blocks at infinity and/or singular pencil
- in engineering practice, often solved by perturbing+inverting R

ARE must be replaced by a system

Lur'e equations

$$\begin{aligned} A^T X + XA + Q &= Y^T Y \\ XB + S &= Y^T Z \\ R &= Z^T Z \end{aligned}$$

(only X needed in practice)

Lur'e equations and deflating subspaces

Deflating subspace formulation

$$\begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \underbrace{\begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}}_{U_X} = \begin{bmatrix} I_n & 0 \\ -X & Y^* \\ 0 & Z^* \end{bmatrix} \begin{bmatrix} -sI + A & B \\ Y & Z \end{bmatrix}$$

$\ker \mathcal{E} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}$ "obvious" deflating subspace ($\lambda = \infty$).

Partial subspace $\begin{bmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{bmatrix} \subseteq U_X \iff$ Partial solution: $\begin{aligned} XV_2 &= V_1 \\ X &= V_1 V_2^+ + \cdots \end{aligned}$

Even Kronecker canonical form

Even Kronecker canonical form [Thompson, '76 & '91] , a powerful tool to analyze Lur'e equations theoretically [Reis, '11]

Canonical form under transformations of the kind $M^T \mathcal{A} M$, $M^T \mathcal{E} M$ (for any M nonsingular)

Plays well with

- deflating subspaces $(\mathcal{A} - s\mathcal{E})U = V(\widehat{\mathcal{A}} - s\widehat{\mathcal{E}})$
- $\mathcal{E}$-neutrality $U^T \mathcal{E} U = 0$ (and similar relations)

Even Kronecker canonical form [Thompson, '76 & '91]

Every even matrix pencil (i.e., $\mathcal{A} = \mathcal{A}^*$, $\mathcal{E} = -\mathcal{E}^*$) can be reduced to a direct sum of the following block types...

Even Kronecker canonical form

$$\left[\begin{array}{ccc|ccc} & & & \lambda - s & 1 & \\ & & & & \lambda - s & 1 \\ & & & & & \lambda - s \\ \hline \bar\lambda + s & & & & & \\ 1 & \bar\lambda + s & & & & \\ & 1 & \bar\lambda + s & & & \end{array}\right]$$

paired eigenvalues $(\lambda, -\bar\lambda)$

$$\begin{bmatrix} & & & s & 1 \\ & & s & 1 & \\ & s & 1 & & \\ s & 1 & & & \\ 1 & & & & \end{bmatrix}$$

eigenvalues at ∞

$$\begin{bmatrix} & & & i\mu - s \\ & & i\mu - s & 1 \\ & i\mu - s & 1 & \\ i\mu - s & 1 & & \end{bmatrix}$$

imaginary eigenvalues $i\mu$

$$\left[\begin{array}{cc|ccc} & & & s & 1 \\ & & s & 1 & \\ \hline & s & & & \\ s & 1 & & & \\ 1 & & & & \end{array}\right]$$

singular blocks

The bad guys

$$\begin{bmatrix} & & & s & 1 \\ & & s & 1 & \\ & s & 1 & & \\ s & 1 & & & \\ 1 & & & & \end{bmatrix} \qquad \left[\begin{array}{cc|ccc} & & & s & 1 \\ & & s & 1 & \\ \hline & s & & & \\ s & 1 & & & \\ 1 & & & & \end{array}\right]$$

eigenvalues at ∞ singular blocks

Singular $R \Leftrightarrow$ nontrivial blocks of one of these two kinds.

Theorem

For all solutions X, U_X contains the first $\frac{\ell-1}{2}$ vectors of each of these Kronecker chains (ℓ=length)

Wong sequences

Pencil generalization of the procedure used to compute Jordan chains/bases [Wong KT, '74] [Berger, Ilchmann, Trenn, '10]

Wong sequence (for $\lambda = \infty$)

$$\mathcal{W}_0 = \{0\}, \qquad \mathcal{W}_{k+1} = \mathcal{E}^{-1}(\mathcal{A}\mathcal{W}_k)$$

(The $\mathcal{W}_i$ are subspaces, and $\mathcal{E}^{-1}$ = preimage)

Switch to even Kronecker form, everything here transforms well

Wong sequences of Kronecker blocks

$$
\begin{array}{c}
\begin{array}{ccccc} & & \mathcal{W}_3 & \mathcal{W}_2 & \mathcal{W}_1 \end{array} \\
\begin{bmatrix} & & & s & 1 \\ & & s & 1 & \\ & s & 1 & & \\ s & 1 & & & \\ 1 & & & & \end{bmatrix}
\end{array}
\qquad
\begin{aligned}
\mathcal{W}_1 &= \operatorname{span}\{e_n\} \\
\mathcal{W}_2 &= \operatorname{span}\{e_{n-1}, e_n\} \\
&\ \vdots
\end{aligned}
$$

Problem: how to force them to stop at half the size of each block?

Idea: that's exactly where they stop being $\mathcal{E}$-neutral!

$$
\begin{array}{c}
\begin{array}{ccccc} & & \mathcal{W}_3 & \mathcal{W}_2 & \end{array} \\
\begin{bmatrix} & & & s & 1 \\ & & s & 1 & \\ & s & 1 & & \\ s & 1 & & & \\ 1 & & & & \end{bmatrix}
\end{array}
$$

$\mathcal{E}$-neutral Wong sequences

Wong sequence (for $\lambda = \infty$)

$$\mathcal{V}_0 = \{0\}, \qquad \mathcal{Z}_k = \mathcal{E}^{-1}(\mathcal{A}\mathcal{V}_k), \qquad \mathcal{V}_{k+1} = \mathcal{V}_k + \mathcal{Z}_k \cap \mathcal{Z}_k^{\mathcal{E}\perp}$$

Theorem

$\mathcal{E}$-neutral Wong sequences are increasing ($\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots$) and stabilize to the space spanned by the first $\frac{\ell+1}{2}$ vectors of each infinite (and singular) chain.

$\mathcal{V}_\infty$ gives a partial solution: $XV_2 = V_1$, $X = V_1 V_2^+ + \cdots$ for some V_1, V_2.

Question How to compute the remaining part?

Projected Lur'e equations

We multiply everything in the Lur'e equations by $\Pi = I - V_2 V_2^+$, and get

Theorem

$\widetilde{X} = \Pi^* X \Pi$ satisfies projected Lur'e equations with

$$\widetilde{A} = \Pi A \Pi, \quad \widetilde{Q} = \Pi^* Q \Pi, \quad \widetilde{B} = \begin{bmatrix} \Pi A V_2 & \Pi B \end{bmatrix},$$
$$\widetilde{S} = \begin{bmatrix} \Pi^* A^* V_1 + \Pi^* Q V_2 & \Pi^* S \end{bmatrix},$$
$$\widetilde{R} = \begin{bmatrix} V_2^* A^* V_1 + V_1^* A V_2 + V_2^* Q V_2 & V_1^* B + V_2^* S \\ B^* V_1 + S^* V_2 & R \end{bmatrix}$$

Projected Lur'e equations

"Projection" $\iff$ zeroing out the critical subspace at infinity
In the right basis,

$$\Pi \begin{bmatrix} 0 & \widetilde{A} - sI & \widetilde{B} \\ \widetilde{A}^* + sI & \widetilde{Q} & \widetilde{S} \\ \widetilde{B}^* & \widetilde{S}^* & \widetilde{R} \end{bmatrix} \Pi^T \cong \begin{bmatrix} 0 & A_1 - sI & B_1 & 0 \\ A_1^* + sI & Q_1 & S_1 & 0 \\ B_1^* & S_1^* & R_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R_1 nonsingular, so we can turn this into a projected Riccati equation

$$\begin{bmatrix} A_{11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

We solve this ARE with Newton-ADI (Lyapack, [Benner, Li, Penzl, '08]).

Problem A_{11} is dense: we must use $\tilde{A} = \Pi A \Pi = (I - V_2 V_2^+) A (I - V_2 V_2^+)$ to preserve sparsity

What happens in ADI

ADI: lots of singular equations with $\Pi A \Pi$:

$$\begin{bmatrix} A_R - zI & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

In fact, if we work with $\Pi A \Pi - zI$ we regularize them for free:

$$\begin{bmatrix} A_R - zI & 0 \\ 0 & -zI \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Further trick: rewrite $(I - V_2 V_2^+) A (I - V_2 V_2^+) x = b$ as extended system

$$\begin{bmatrix} A & V_2 & \Pi A V_2 \\ V_2^+ A & I & 0 \\ V_2^+ & 0 & I \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}$$

Preserves sparsity, now we can use sparse LU

To sum up

Algorithm

1. Compute $\mathcal{V}_\infty$ "critical subspace" using $\mathcal{E}$-neutral Wong sequences
2. Compute coefficients $\widetilde{B}$, $\widetilde{R}$, $\widetilde{S}$ of the projected equation, and sparse representations of $\widetilde{A} = \Pi A \Pi$, $\widetilde{Q} = \Pi^* Q \Pi$
3. Use Newton-ADI to solve the projected Riccati equation for $\widetilde{X}$. Use extended matrix approach for solvers.
4. Assemble solution $X = V_1 V_2^+ + \widetilde{X}$

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On combining deflation and iteration to low-rank approximate solution of Lur'e equations
U Hamburg preprint, submitted.

Example I

Lur'e equations from positive real lemma

Demo system `demo-r1` in Lyapack (heat equation on the square)

	`demo-r1`
n	2500
m	1
rank decisions accuracy	1.6×10^{-16}
infinite chains	$1 \times$ length 3
singular chains	0
rank of $X^{(1)}$	24
rank of $X - X^{(1)}$	23
no. of Newton steps needed	4
avg. ADI itns per Newton step	37.25
relative residual	2.6×10^{-15}
deviation from stability	-1.8×10^{-15}
CPU time	$17\,\mathrm{s}$

Example II

Lur'e equations from positive real lemma

Demo system `demo-r3` in Lyapack (rail profile)

	`demo-r3`
n	821
m	6
rank decisions accuracy	6.5×10^{-16}
infinite chains	$6 \times$ length 3
singular chains	0
rank of $X^{(1)}$	138
rank of $X - X^{(1)}$	130
no. of Newton steps needed	7
avg. ADI itns per Newton step	36.857
relative residual	5.5×10^{-15}
deviation from stability	-1.3×10^{-08}
CPU time	$65\,\mathrm{s}$

Example II

Lur'e equations from positive real lemma
Demo system `demo-r3` in Lyapack (rail profile)

Thanks for your attention! Questions?

	`demo-r3`
rank of $X - X^{(1)}$	130
no. of Newton steps needed	7
avg. ADI itns per Newton step	36.857
relative residual	5.5×10^{-15}
deviation from stability	-1.3×10^{-08}
CPU time	$65\,\mathrm{s}$