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Randomized Methods for the Algebraic Eigenvalue Problem

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Preface

The use of randomization in design and analysis of algorithms as well as in computational complexity has been very popular in the last few years. Randomized algorithms have been used in a number of different fields in theoretical computer science and applied mathematics. The adoption of randomization for algorithm design has produced a lot of very interesting results nowadays, especially for discrete problems. However, randomized algorithms were originally proposed to solve continuous problems. That was during the Manhattan project in 1949 where Metropolis and Ulam coined the name of Monte Carlo method and presented a randomized algorithm for multivariate integration.

This thesis deals with the study of randomized algorithms for estimating eigenvalues and eigenvectors. This is a new field of application of randomized methods. In particular, we consider the problem of estimating the largest eigenvalue and a corresponding eigenvector of a symmetric matrix. For this specific problem, iterative methods are generally preferred to methods based on matrix factorization.

The goal of this thesis is to study if the use of randomized techniques helps for the problem of estimating an eigenpair. We tackle this problem following the two different approaches.

The first consists of studying randomized algorithms whose cost is less than that of an efficient deterministic method. We propose an algorithm that computes an approximation of the largest eigenvalue, and whose parallel cost is lower than that of the best available deterministic implementation of the power method.

The second is the error analysis of very well known methods such as the power and Lanczos methods, when the starting vector is chosen randomly. The interest for this problem arises since the above mentioned methods are not convergent to the quantities we want to estimate in the worst deterministic case. In practical cases, the choice of the initial vector is done at random, and by analyzing the randomized error, we can determine the actual speed of convergence. Together with the study of the randomized error for a complete class of error parameters, we study the possibility of bounding the randomized error by a quantity that approaches zero as the number of iterations increases, and that does not depend on the matrix. This is a very attractive question: indeed, if we are able to bound the error by a value that does not depend on the matrix, we can give an a priori estimate of the number of steps that guarantees the error to be lower than a given threshold. Clearly, this is not possible in the deterministic case, since there are matrices for which these methods, for a fixed starting vector, do not converge to the largest eigenvalue and to a corresponding eigenvector. In the randomized setting, we prove that the problem of estimating an eigenvector is more difficult than that of estimating an eigenvalue. In fact, while for the eigenvalue case it is possible to give bounds independent of the distribution of the eigenvalues, for the eigenvector this is not the case. This means that randomization helps for eigenvalue estimate but not for eigenvector.

This thesis is organized as follows. Chapter 1 contains a general introduction to the subject as well as the definition of the randomized error in the sense of \mathcal{L}_p , $p \in [1, +\infty]$.

In Chapter 2 we present results about the estimate of the largest eigenvalue and of a corresponding

eigenvector by the power method with a random start. In particular, we give distribution-free bounds for eigenvalue approximation and we show that, on the contrary, for eigenvector estimate we cannot get bounds independent of the eigenvalues of the matrix. In fact, there are matrices for which the method does not converge to the desired eigenvector, for a random starting vector. We achieve also upper and lower bounds that do depend on the spectrum of the matrix to which we apply the method. These bounds are asymptotically optimal, as it can be proven by analyzing the asymptotic randomized error. Moreover, we show that the speed of convergence depends on the particular error parameter and on the multiplicity of the largest eigenvalue.

Chapter 3 presents similar results for Lanczos method and for other polynomial methods. We prove a general negative results that holds for any polynomial method for eigenvector estimate. More precisely, we show that, for any polynomial method, there exists a symmetric positive definite matrix for which the method fails in estimating an eigenvector belonging to the largest eigenvalue when less than n steps are performed. This should be contrasted with eigenvalue estimate. For this problem we can indeed give bounds that do not depend on the eigenvalues of the matrix, and that approaches zero when the number of iteration increases. For Lanczos method we also give upper bounds on the error that depend on the distribution of the eigenvalues.

Both power and Lanczos methods have been tested on several matrices with different eigenvalue distributions. The results of these numerical tests are reported in the last section of each chapter.

In Chapter 4 we propose a fully randomized algorithm. The parallel cost of this algorithm is lower than the cost of the best deterministic implementation of the power method on a CREW-PRAM. The crucial part of our analysis is the study of the probabilistic error.

Chapter 5 contains the conclusions and some final remarks.

Chapter 1

Introduction

In this thesis we consider the problem of estimating the largest eigenvalue and a corresponding eigenvector of a symmetric matrix. There are many algorithms for approximating eigenvalues of $n \times n$ symmetric matrices. For a complete list of methods see for example [Par80] [Saa92] [Wil65]. Algorithms based on factorizations of A cost $\Theta(n^3)$ operations, and are not convenient when we want to estimate only an eigenpair since they compute unnecessarily the complete spectrum of the matrix. Moreover, these algorithms do not take advantage of the possible sparsity of the matrix. Methods based on Krylov information are more convenient for our purposes since matrix-vector products can be performed cheaply, and we can exploit the structure of the matrix.

Krylov information can be viewed as a $n \times k$ matrix, whose columns are the matrix-vector multiplications of the first $k - 1$ powers of the matrix A times a unit vector \mathbf{b} . Among algorithms that use Krylov information we find the well known power method and the far superior Lanczos method. These methods are iterative and they need a starting vector. It is well known that both these methods fails to converge to the largest eigenvalue and to a corresponding eigenvector if the starting vector is orthogonal to the eigenspace of the largest eigenvalue λ_1 . As we will see in Section 1.2 this negative result can be easily extended to all Krylov methods (see Section 1.1 for the definition) as long as the starting vector \mathbf{b} is chosen deterministically and independently on the matrix A . Moreover, also if Krylov information is replaced by any other k matrix-vector multiplications, the problem is still unsolvable in the deterministic setting as long as the number of iterations k is less than n (see Section 1.2).

This motivates the choice of a random starting vector \mathbf{b} and the study of the “randomized error”. This means to analyze the behavior of these methods when the initial vector is distributed according to a probability measure μ . This approach has been taken in [KW92a], [KW94], [dCM96] for the approximation to the largest eigenvalue and in [dCo96], [LW96] and [dCM96] for the approximation to a normalized eigenvector.

It is well known that when the starting vector is chosen deterministically, the speed of convergence of the power and Lanczos methods, depends dramatically on the ratio between the first two largest eigenvalues. Kuczyński and Woźniakowski [KW92a] analyzed the randomized error for estimating the largest eigenvalue and they gave bounds on the randomized error that do not depend on the distribution

of the eigenvalues. Distribution-free bounds can be used to determine the number of steps that, for every matrix, guarantees the randomized error to be lower than a positive threshold ε . In this thesis we refined and generalized distribution-free as well as distribution dependent randomized error bounds proposed in the literature.

For the approximation of an eigenvector it is possible to prove that it is not possible to give bound of the error that do not depend on the eigenvalues. We first prove this for the power method (Chapter 2) and then for a generic polynomial method (Chapter 3).

In the last chapter of this thesis we propose a parallel randomized version of the power method for eigenvalue estimate. The approach is different from that adopted in the other chapters. In fact, while in Chapter 2 and 3 the use of randomization is limited at the first step of the algorithms, in Chapter 4 we develop a fully randomized algorithm, in the sense that randomization is used during the algorithm execution to make non-deterministic choices.

The thesis is organized as follows. This chapter is devoted to introduce the problem and the terminology. Chapter 2 presents results about the randomized error of power method with a random start. In Chapter 3, we bound the randomized error for eigenvalue and eigenvector estimate by Lanczos method. Chapter 4 contains a randomized parallel algorithm for estimating the largest eigenvalue of a matrix. This algorithm is based on the power method and its parallel cost as well as the probabilistic error are analyzed. Chapter 5 contains some conclusions.

1.1 Definitions and Notation

Let A be an $n \times n$ symmetric matrix. Without loss of generality we assume that A is positive definite. Indeed, if β is a constant, such that $\beta > \|A\|$, then the matrix $A + \beta I$ is positive definite and its eigenvalues are in a one-to-one correspondence with those of A . Here one can choose an arbitrary norm, but it is convenient to use $\|\cdot\|_1$ or $\|\cdot\|_\infty$ to have an easy upper bound of $\|A\|$. In this thesis $\|\cdot\|$ denotes the Euclidean norm unless otherwise specified.

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ denote the eigenvalues of A and let $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ be the corresponding orthonormal eigenvectors.

Assume $r, r < n$, is the multiplicity of the largest eigenvalue λ_1 and \mathbf{Z}_1 the eigenspace corresponding to λ_1 , i.e.

$$\mathbf{Z}_1 = \text{span} \{ \mathbf{z} : A\mathbf{z} = \lambda_1\mathbf{z} \}.$$

We consider the problem of approximating λ_1 and a vector in \mathbf{Z}_1 . This problem is often solved using iterative methods. As already pointed out, when we want to compute only few eigenvalues and eigenvectors, iterative methods are preferable to QR or QL algorithms for their lower computational cost. Without loss of generality (see Section 1.2) we restrict ourselves to algorithms that use Krylov information, that is, the set $N_k(A, \mathbf{b})$ given by

$$N_k(A, \mathbf{b}) = \{ \mathbf{b}, A\mathbf{b}, \dots, A^{k-1}\mathbf{b} \},$$

where \mathbf{b} is a vector with unit norm. We define the Krylov subspace as

$$\mathcal{K}_k = \text{span} \left\{ \mathbf{b}, A\mathbf{b}, \dots, A^{k-1}\mathbf{b} \right\},$$

and we call *Krylov subspace methods* or simply Krylov methods, those methods that extract approximation from the Krylov subspace.

A *polynomial algorithm* [LW96] is an algorithm that produces at the k -th step a vector $\mathbf{u}_k = \mathbf{u}(A, \mathbf{b}, k)$ such that \mathbf{u}_k belongs to \mathcal{K}_k and $\|\mathbf{u}_k\| = 1$. Hence, \mathbf{u}_k can be written as

$$\mathbf{u}_k = c_0\mathbf{b} + c_1A\mathbf{b} + \dots + c_{k-1}A^{k-1}\mathbf{b} = P(A)\mathbf{b},$$

where P is a polynomial of degree at most $k - 1$ such that $\|\mathbf{u}_k\| = 1$. Within this class the most widely used algorithms are the power method and the Lanczos method.

We use the following notation. For any polynomial method \mathcal{M} , $\xi_k^{\mathcal{M}} = \xi_k^{\mathcal{M}}(A)$ and $\mathbf{u}_k^{\mathcal{M}} = \mathbf{u}_k^{\mathcal{M}}(A)$ denote the approximation to λ_1 and $\mathbf{z} \in \mathbf{Z}_1$ returned by the method \mathcal{M} after k steps.

We can define the power method as

$$\left\{ \begin{array}{l} \mathbf{x}_k = A\mathbf{x}_{k-1}, \\ \mathbf{u}_k^{\text{Pow}} = \frac{A\mathbf{u}_{k-1}^{\text{Pow}}}{\|A\mathbf{u}_{k-1}^{\text{Pow}}\|}, \\ \xi_k^{\text{Pow}} = \frac{\mathbf{x}_{k-1}^T A \mathbf{x}_{k-1}}{\mathbf{x}_{k-1}^T \mathbf{x}_{k-1}}, \end{array} \right. \quad (1.1)$$

where $\mathbf{u}_0^{\text{Pow}} = \mathbf{x}_0 = \mathbf{b}$ is the initial vector.

For the Lanczos method we have

$$\xi_k^{\text{Lan}} = \max \left\{ \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}, \mathbf{x} \in \mathcal{K}_k \right\} \quad (1.2)$$

and $\mathbf{u}_k^{\text{Lan}}$ is the unit norm vector such that

$$R_k A \mathbf{u}_k^{\text{Lan}} = \xi_k^{\text{Lan}} \mathbf{u}_k^{\text{Lan}},$$

where R_k is the orthogonal projector onto \mathcal{K}_k (for more details see [LW96]). Although this definition cannot be used in practice, there exists a very efficient formulation of the method that makes Lanczos method one of the most widely used method for the computation of extremal eigenvalues (see for example [Saa92] [Wil65] [BCM88]).

The value ξ_k^{Lan} can be seen as the “best” approximation to λ_1 we can obtain using vectors in \mathcal{K}_k . A similar property does not hold for $\mathbf{u}_k^{\text{Lan}}$. For this reason we introduce the *best polynomial method* \mathcal{B} for eigenvector estimate. At step k this algorithm returns the vector $\mathbf{u}_k^{\mathcal{B}}$, with $\|\mathbf{u}_k^{\mathcal{B}}\| = 1$, such that

$$\inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\mathcal{B}} - \mathbf{v}\| = \inf_{\substack{\mathbf{u} \in \mathcal{K}_k \\ \|\mathbf{u}\|=1}} \inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u} - \mathbf{v}\|. \quad (1.3)$$

Although the computation of $\mathbf{u}_k^{\mathcal{B}}$ is not feasible, the method \mathcal{B} has a significant theoretical interest since important properties of Lanczos method can be derived from the study of the method \mathcal{B} (see for example [CL75], [Saa80]).

It can be shown that all polynomial methods fail to converge to λ_1 and to a corresponding eigenvector when the starting vector \mathbf{b} is orthogonal to the eigenspace \mathbf{Z}_1 . In fact, in that case, the starting vector \mathbf{b} has all zero component in the direction of the eigenspace \mathbf{Z}_1 , and we cannot recover information about λ_1 and a corresponding eigenvector from $N_k(A, \mathbf{b})$. Moreover, the analysis of the deterministic speed of convergence (assuming $\mathbf{b} \not\perp \mathbf{Z}_1$) shows that the convergence depends on the distribution of the eigenvalues (see [Par80], [Saa80], [Saa92], [Wil65]).

In [dCo96], [dCM96], [KW92a] and [LW96] is analyzed the behavior of these methods assuming the starting vector \mathbf{b} to be randomly chosen with uniform distribution μ on the n -dimensional sphere of radius one, $\mu(\{\mathbf{b} \in \mathbb{R}^n : \|\mathbf{b}\| = 1\}) = 1$. The convenience of this measure, as it has been underlined in [KW92a], is due to the facts that the values $\xi_k^{\mathcal{M}}$ and $\mathbf{u}_k^{\mathcal{M}}$ are independent of the norm of the starting vector \mathbf{b} and are invariant for multiplication by orthogonal matrices. Hence, it is wise to use a distribution that is concentrated on the unit sphere of \mathbb{R}^n and that is orthogonal invariant as the measure μ .

In this thesis we analyze the *randomized error in the \mathcal{L}_p sense*, defined as follows. Let $1 \leq p < +\infty$, and let $S_n = \{\mathbf{b} \in \mathbb{R}^n : \|\mathbf{b}\| = 1\}$. The randomized error in the \mathcal{L}_p case for the approximation to an eigenvector is defined as

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{M}}, A, p) = \left(\int_{S_n} \inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\mathcal{M}}(b) - \mathbf{v}\|^p \mu(db) \right)^{1/p}, \quad (1.4)$$

that is the \mathcal{L}_p norm of the minimum distance between the vector $\mathbf{u}_k^{\mathcal{M}}(b)$ and the eigenspace \mathbf{Z}_1 . The randomized error in the sense of \mathcal{L}_p for the approximation to the largest eigenvalue λ_1 is

$$e^{\text{ran}}(\xi_k^{\mathcal{M}}, A, p) = \left(\int_{S_n} \left| \frac{\lambda_1 - \xi_k^{\mathcal{M}}(b)}{\lambda_1} \right|^p \mu(db) \right)^{1/p}, \quad (1.5)$$

i.e. the \mathcal{L}_p norm of the relative error of the method \mathcal{M} for the approximation to λ_1 .

The study of the randomized error for all possible p emphasizes the importance of the choice of the error parameter for the convergence of the method. In fact, it turns out that the value of the parameter p and, in particular, its relation with the multiplicity r of the largest eigenvalue λ_1 , determines the speed of convergence of the randomized algorithms.

In the thesis we report also some numerical tests in order to evaluate the sharpness of the bounds provided. Of course, we had to simulate the randomized behavior of the power and Lanczos methods with vectors uniformly distributed over the unit sphere of \mathbb{R}^n . Such vectors can be generated in accordance with a method first proposed in [Bro56]. It can be described as follows.

1. For every component generate uniformly and independently two random numbers U_1, U_2 in $[0, 1]$.
2. For every component compute $Y = \sqrt{-2 \ln(U_1)} * \cos(2\pi U_2)$. Then Y is normally distributed over $(0, 1)$.

3. Let Y_1, Y_2, \dots, Y_n be n independent number normally distributed over $(0, 1)$ generated in accordance with previous steps. Then the vector $X = (Y_1/\|Y\|, Y_2/\|Y\|, \dots, Y_n/\|Y\|)$ is uniformly distributed over the unit sphere S_n .

In the rest of the thesis we make use of the following notation. c_m denotes the Lebesgue measure of the unit ball over \mathbb{R}^m , that is

$$c_m = \frac{\pi^{m/2}}{\Gamma(m/2 + 1)}, \quad (1.6)$$

see [8.310, 1] of [GR94] for the definition of the gamma function $\Gamma(x)$. We will also use the following relation between the beta and gamma functions

$$B(i, j) = 2 \int_0^1 t^{2i-1} (1-t^2)^{j-1} dt = \frac{\Gamma(i) \Gamma(j)}{\Gamma(i+j)}. \quad (1.7)$$

We will denote by $F(a, b; c; x)$ the hypergeometric function, see [9.10] of [GR94] for the definition and the properties of this function. $T_k(x)$ is the k -th Chebyshev polynomial of first kind, that can be recursively defined as follows:

$$T_k(t) = 2tT_{k-1}(t) - T_{k-2}(t), \quad (1.8)$$

where $T_2(t) = t$ and $T_1(t) = 1$. Chebyshev polynomials are very powerful because of this theorem.

Theorem 1.1.1 (Minimax Theorem) *Let $[\alpha, \beta]$ be a non-empty interval in \mathbb{R} , and let γ be a real number such that $\gamma \geq \beta$. We have*

$$\min_{P \in \mathcal{P}_k, P(\gamma)=1} \max_{t \in [\alpha, \beta]} |P(t)| = \frac{1}{\left| T_k \left(1 + 2 \frac{\gamma - \beta}{\beta - \alpha} \right) \right|}.$$

□

Let B_n denote the unit ball of \mathbb{R}^n and let $f : B_n \rightarrow \mathbb{R}$ be a measurable function such that $f(\mathbf{b}) = f(\alpha \mathbf{b})$ for every $\alpha > 0$, that is f does not depend on the norm of \mathbf{b} . Moreover assume f does not depend on the signs of \mathbf{b} , that is $f(s_1 b_1, s_2 b_2, \dots, s_n b_n) = f(b_1, b_2, \dots, b_n)$, for $s_i \in \{-1, 1\}$. For any of such functions we have

$$\int_{S_n} f(\mathbf{b}) \mu(db) = \frac{1}{c_n} \int_{B_n} f(\mathbf{b}) db. \quad (1.9)$$

Equality (1.9) is often used in the rest of the thesis.

1.2 Some Remarks about Krylov Methods

In this section we present some interesting considerations about algorithms based on Krylov information. These results are due to Kuczyński and Woźniakowski in [KW92a].

We already pointed out that, for methods as the power or Lanczos methods, the choice of the initial vector \mathbf{b} is very crucial since if \mathbf{b} is orthogonal to the eigenspace \mathbf{Z}_1 these methods are not convergence to the quantities we would like to estimate. It is natural to ask if there exists an algorithm \mathcal{M} based on Krylov information for which for all symmetric positive matrices and for some $\varepsilon \in [0, 1)$ we have

$$\left| \frac{\xi_k^{\mathcal{M}} - \lambda_1}{\lambda_1} \right| \leq \varepsilon.$$

Unfortunately this is not the case ¹. In order to prove this let us denote by $d = d(A, \mathbf{b}, k)$ the dimension of the space \mathcal{K}_k . Of course $1 \leq d \leq \min\{k, n\}$. Assume first $d \leq n - 1$. In this case, we claim that there exists a positive definite matrix \bar{A} such that $\xi_k^{\mathcal{M}}(A) = \xi_k^{\mathcal{M}}(\bar{A})$ and

$$\left| \frac{\xi_k^{\mathcal{M}}(\bar{A}) - \lambda_1(\bar{A})}{\lambda_1(\bar{A})} \right| > \varepsilon, \quad (1.10)$$

that is, \mathcal{M} fail to approximate $\lambda_1(\bar{A})$. To prove this we construct a matrix \bar{A} with the above properties. Let $\bar{A} = A + \alpha \mathbf{u}\mathbf{u}^T$, where α is a positive constant and \mathbf{u} is a nonzero vector orthogonal to \mathcal{K}_k . Since $d \leq n - 1$ such vector exists and moreover it is easy to prove by induction that $N_k(A, \mathbf{b}) = N_k(\bar{A}, \mathbf{b})$. This implies $\xi_k^{\mathcal{M}}(A) = \xi_k^{\mathcal{M}}(\bar{A})$. Observe that

$$\text{trace}(\bar{A}) = \text{trace}(A) + \alpha \|\mathbf{u}\|^2,$$

and it goes to infinity as α goes to infinity. This means that $\lambda_1(\bar{A})$ goes to infinity. We have

$$\left| \frac{\xi_k^{\mathcal{M}}(\bar{A}) - \lambda_1(\bar{A})}{\lambda_1(\bar{A})} \right| = \left| \frac{\xi_k^{\mathcal{M}}(A) - \lambda_1(\bar{A})}{\lambda_1(\bar{A})} \right| \rightarrow 1, \quad \text{as } \alpha \rightarrow +\infty.$$

Hence there exists an α such that (1.10) is satisfied as claimed. We can notice that for large α , the largest eigenvalue of \bar{A} is very close to α , while the information $N_k(\bar{A}, \mathbf{b})$ is almost independent of \mathbf{u} , so it cannot well approximate $\lambda_1(\bar{A})$ or a corresponding eigenvector with a relative error at most ε .

This proves that among the class of algorithm that uses Krylov information with $k \leq n - 1$ there always exists a symmetric positive definite matrix that shares the same information with A but for which it is impossible to estimate the largest eigenvalue with relative error at most ε . If $d = n$ then \mathcal{K}_k spans all the space, then we have enough information to compute the largest eigenvalue.

If more than n steps are performed, we cannot always guarantee $d = n$, this happens for example when the starting vector \mathbf{b} is an eigenvector of A . In this case $d(A, \mathbf{b}, k) = 1$ for every k and we cannot compute $\lambda_1(A)$ with the required accuracy.

Note that in order to have $d = n$ we need the starting vector \mathbf{b} to have nonzero component in the direction of every eigenvector \mathbf{z}_i for $i = 1, 2, \dots, n$. This cannot be guaranteed for all the possible vectors \mathbf{b} , but for a vector chose uniformly over the unit sphere of \mathbb{R}^n , this holds with probability 1 if and only if A has n distinct eigenvalues. This observation allows one to expect that, even if, in the deterministic setting, there are vectors for which methods based on Krylov information are not convergent, in the randomized setting this problem can be solved.

So far we proved that Krylov information is not strong enough for computing an ε -approximation to the largest eigenvalue and to a corresponding eigenvector. It is natural to ask if there are other methods that using matrix-vector multiplications lead to positive results. Using matrix-vector multiplications we can compute $N(A) = [A\mathbf{q}_1, A\mathbf{q}_2, \dots, A\mathbf{q}_k]$, where $\mathbf{q}_1 = \mathbf{b}$ and where each \mathbf{q}_i is an arbitrary function of the vectors $A\mathbf{q}_1, A\mathbf{q}_2, \dots, A\mathbf{q}_{i-1}$. Unfortunately it can be proved [TWW88] that the minimal number of matrix-vector multiplications to compute an ε -approximation of λ_1 for all the positive definite matrix is still n .

¹We report the proof only for the approximation to the largest eigenvalue but it is straightforward to extend these results for eigenvectors estimate.

Chapter 2

The Power Method with a Random Start

2.1 Introduction and Preliminaries

In this chapter we consider the problem of estimating the largest eigenvalue and a corresponding eigenvector of a positive definite matrix by the power method with a random start. We already underlined that this method as well as the other methods based on Krylov information may not converge to the desired quantities when the starting vector is chosen deterministically. It happens if the starting vector has zero component in the direction of the eigenspace corresponding to the largest eigenvalue. In this case we may have convergence to the second largest eigenvalue and to a corresponding eigenvector. Another problem that arises using the power method is that the speed of convergence depends dramatically on the ratio between the two largest eigenvalues (see [GL89], [Par80], [Saa92], [Wil65]). In particular, if we denote by r the multiplicity of λ_1 , we have that the speed of convergence for the approximation of λ_1 is given by $(\lambda_{r+1}/\lambda_1)^{2k}$ while the rate of convergence for estimating a dominant eigenvector is $(\lambda_{r+1}/\lambda_1)^k$. This shows that, when the eigenvalues are badly separated, the convergence is very slow. With the study of the randomized error of the method we aim to explore also if it is possible to get bounds of the randomized error that does not depend on the matrix to which the method is applied. Surprisingly the power method behaves differently for the approximation to λ_1 and to an eigenvector of \mathbf{Z}_1 .

We briefly comment on related work on approximate computation of eigenpairs. The idea of using random starting vectors for the power method can be found in the paper of Shub [Shu86]. Shub applies the power method to the matrix e^{-A} , and approximates an eigenvector of A which is *not* necessarily a largest eigenvector. Although for this problem the power method is globally convergent, the random start is used to improve efficiency. Shub shows, however, that even for $n = 2$ there are matrices for which this problem is very hard. In this thesis we apply the power method to the matrix A and we are only interested in approximating a largest eigenvector.

Kostlan [Kos88] studies the randomized performance of the power method. In particular, in that

paper he bounds the number of steps that allows the error to be lower than a fixed threshold ε . We discuss those bounds in Section 2.5 comparing with the bounds proposed in this thesis.

Wright [Wri89] and Kostlan [Kos91] analyzed the problem of approximating a largest eigenvector by the power method in a different setting. They considered the average case setting over a class of matrices, whereas we consider the randomized setting. In particular, they estimate the average time needed for computing a vector whose relative distance from the eigenspace of largest eigenvectors is less than ε . In our paper the matrix is fixed while the starting vector is chosen at random.

This chapter is organized as follows. Section 2.2 is devoted to the study of the randomized error for eigenvalue approximation. In particular we present a sharp upper and lower bound for the \mathcal{L}_1 case (due to Kuczyński and Woźniakowski in [KW92a]) that does not depend on the distribution of the eigenvalues. Moreover we show that if we know the multiplicity of λ_1 distribution free bounds for the \mathcal{L}_p case can be given. Then we give upper and lower bounds for eigenvalue approximation that do depend on the particular matrix. In Section 2.3 we analyze the randomized error for eigenvector approximation. We first show that it is impossible to achieve distribution free bounds and then we show sharp upper and lower bounds. In Section 2.5 we discuss the bounds provided in the chapter. Section 2.6 contains numerical tests that allows us to explain and comment the results of this chapter.

The results of this chapter have been presented also in [dCo96] and in [dCM96].

The main technical difficulties are in the computation of integrals of the kind

$$\int_{S_n} \left(\frac{a \sum_{i=q+1}^n b_i^2}{\sum_{i=1}^q b_i^2 + a \sum_{i=q+1}^n b_i^2} \right)^s \mu(db).$$

over the n -dimensional sphere of radius one. In order to simplify the exposition, the solution of this integrals as well as tight upper and lower bounds are given below.

Let \mathbf{b} be a real n -vector, such that $\|\mathbf{b}\|^2 = \sum_{i=1}^n b_i^2$. Let $a \in (0, 1)$, $s \in [1/2, +\infty)$, and q an integer such that $1 \leq q < n$. We compute the following integral

$$I = \int_{S_n} \left(\frac{a \sum_{i=q+1}^n b_i^2}{\sum_{i=1}^q b_i^2 + a \sum_{i=q+1}^n b_i^2} \right)^s \mu(db). \quad (2.1)$$

Setting the parameters s and q conveniently one can get bounds on the randomized error estimate for eigenvalue and eigenvector approximation. We have the following lemma.

Lemma 2.1.1 *Let $a \in (0, 1)$, $s \in [1/2, +\infty)$, and let q be an integer $1 \leq q < n$. Then the integral in equation (2.1) is given by*

$$I = a^s \frac{\Gamma((n-q)/2 + s) \Gamma(n/2)}{\Gamma((n-q)/2 + 1) \Gamma(n/2 + s)} F\left(\frac{n-q}{2} + s, s; \frac{n}{2} + s; 1 - a\right).$$

Proof. Due to equation (1.9), it is possible to rewrite this integral as an integral over the unit ball B_n . We have

$$I = \frac{1}{c_n} \int_{B_n} \left(\frac{a \sum_{i=q+1}^n b_i^2}{\sum_{i=1}^q b_i^2 + a \sum_{i=q+1}^n b_i^2} \right)^s db. \quad (2.2)$$

Rewriting (2.2) as an integral over the unit ball B_{n-q} and the q -dimensional ball B'_q of radius $d = \left(1 - \sum_{i=q+1}^n b_i^2\right)^{1/2}$, we get

$$I = \frac{a^s}{c_n} \int_{B_{n-q}} \left(\sum_{i=q+1}^n b_i^2 \right)^s \int_{B'_q} \frac{1}{\left(\sum_{i=1}^q b_i^2 + a \sum_{i=q+1}^n b_i^2 \right)^s} db. \quad (2.3)$$

Applying formula [4.642] of [GR94] we reduce the second integral of (2.3) to a one-dimensional integral. We get

$$I = a^s \frac{q c_q}{c_n} \int_{B_{n-q}} \left(\sum_{i=q+1}^n b_i^2 \right)^s \int_0^d \frac{t^{q-1}}{\left(t^2 + a \sum_{i=q+1}^n b_i^2 \right)^s} dt db. \quad (2.4)$$

Changing variables by setting $t^2 = u$, we get

$$\begin{aligned} I &= a^s \frac{q c_q}{2 c_n} \int_{B_{n-q}} \left(\frac{\sum_{i=q+1}^n b_i^2}{a \sum_{i=q+1}^n b_i^2} \right)^s \int_0^{d^2} \frac{u^{q/2-1}}{\left(u/(a \sum_{i=q+1}^n b_i^2) + 1 \right)^s} du db \\ &= \frac{q c_q}{2 c_n} \int_{B_{n-q}} \frac{2}{q} \left(1 - \sum_{i=q+1}^n b_i^2 \right)^{q/2} F \left(s, \frac{q}{2}; \frac{q}{2} + 1; -\frac{1 - \sum_{i=q+1}^n b_i^2}{a \sum_{i=q+1}^n b_i^2} \right) db, \end{aligned} \quad (2.5)$$

where equation (2.5) is obtained applying formula [3.194,1] on [GR94]. We apply again formula [4.642] to equation (2.5). We have

$$I = \gamma \int_0^1 b^{n-q-1} (1-b^2)^{q/2} F \left(s, \frac{q}{2}; \frac{q}{2} + 1; -\frac{1-b^2}{a b^2} \right) db, \quad (2.6)$$

where $\gamma = c_q(n-q)c_{n-q}/c_n$. Changing variables by setting $z = (1-b^2)/(a b^2)$ equation (2.6) becomes

$$I = \frac{\gamma}{2 a^{(n-q)/2}} \int_0^{+\infty} \frac{z^{q/2}}{(z+1/a)^{n/2+1}} F \left(s, \frac{q}{2}; \frac{q}{2} + 1; -z \right) dz.$$

Applying formula [7.512, 10] of [GR94] and the transformation formula of hypergeometric function [9.131, 1] of [GR94] we get

$$\begin{aligned} I &= \frac{\gamma}{2 a^{(n-q)/2}} \frac{\Gamma(q/2+1) \Gamma((n+2s-q)/2) \Gamma(n/2)}{\Gamma(n/2+1) \Gamma(n/2+s)} F \left(\frac{n+2s-q}{2}, \frac{n}{2}; \frac{n}{2} + s; 1 - \frac{1}{a} \right) \\ &= a^s \frac{\Gamma((n+2s-q)/2) \Gamma(n/2)}{\Gamma((n-q)/2+1) \Gamma(n/2+s)} F \left(\frac{n+2s-q}{2}, s; \frac{n}{2} + s; 1 - a \right). \end{aligned}$$

□

Lemma 2.1.1 gives the solution of the integral (2.1). Unfortunately, because of the hypergeometric function, it is difficult to estimate what is the order of convergence to zero of I as $a \rightarrow 0$. However, note that, since $a > 0$, the hypergeometric series that defines the hypergeometric function is convergent.

We prove the following upper bound

Lemma 2.1.2 *Under the same hypothesis of lemma 2.1.1 we have*

$$I \leq \begin{cases} a^s \frac{\Gamma((q-2s)/2) \Gamma((n-q+2s)/2)}{\Gamma(q/2) \Gamma((n-q)/2)} & \text{for } 2s < q \\ a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-2s)/2)} \ln\left(\frac{1}{a}\right) + a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-2s)/2)} \left(2 + \frac{2}{n}\right) & \text{for } 2s = q \\ a^{q/2} \frac{\Gamma(n/2) \Gamma((2s-q)/2)}{\Gamma((n-q)/2) \Gamma(s)} & \text{for } s > q \end{cases} \quad (2.7)$$

Proof. Consider first the case $2s < q$. Let $\|b\|_q^2 = \sum_{i=1}^q b_i^2$ and let $t_i = b_i/(1-\|b\|^2)^{1/2}$ for $i = q+1, \dots, n$ with $\|t\|_{n-q}^2 = \sum_{i=q+1}^n t_i^2$. From (2.2), we have

$$I = \frac{a^s}{c_n} \int_{B_q} \int_{B_{n-q}} \frac{\|t\|_{n-q}^{2s} (1 - \|b\|_q^2)^{(n-q)/2+s}}{\left(\|b\|_q^2 + a\|t\|_{n-q}^2 (1 - \|b\|_q^2)\right)^s} dt db.$$

Applying twice formula [4.642] of [GR94] we reduce the previous integral to a two-dimensional integral. We get

$$I = a^s \gamma \int_0^1 \int_0^1 \frac{t^{n+2s-q-1} b^{q-1} (1-b^2)^{(n-q)/2+s}}{(b^2 + a t^2 (1-b^2))^s} db dt, \quad (2.8)$$

where $\gamma = qc_q(n-q)c_{n-q}/c_n$. Since $b^2 + at^2(1-b^2) \geq b^2$, from (2.8) we get

$$\begin{aligned} I &\leq a^s \gamma \int_0^1 t^{n+2s-q-1} dt \int_0^1 b^{q-2s-1} (1-b^2)^{(n+2s-q)/2} db \\ &= a^s \frac{\gamma}{2(n-q+2s)} B\left(\frac{q-2s}{2}, \frac{n-q+2s}{2} + 1\right). \end{aligned} \quad (2.9)$$

Substituting into (2.9) the expression for γ and simplifying, we have

$$I \leq a^s \frac{\Gamma((q-2s)/2) \Gamma((n-q+2s)/2)}{\Gamma(q/2) \Gamma((n-q)/2)}.$$

Consider now the case $2s = q$. Equation (2.2) can be rewritten with respect to the ball B_{n-q} and to the ball $B'_q = \{\mathbf{b} : \sum_{i=1}^q b_i^2 \leq 1 - \sum_{i=q+1}^n b_i^2\}$. We have

$$I = \frac{a^{q/2}}{c_n} \int_{B_{n-q}} \|b\|_{n-q}^q \int_{B'_q} \frac{1}{\left(\|t\|_q^2 + a\|b\|_{n-q}^2\right)^{q/2}} dt db,$$

where $\|b\|_{n-q}^2 = \sum_{i=q+1}^n b_i^2$ and $\|t\|_q^2 = \sum_{i=1}^q b_i^2$. Applying twice formula [4.642] of [GR94], we get

$$I = a^{q/2} \frac{qc_q(n-q)c_{n-q}}{c_n} \int_0^1 b^{n-1} \int_0^{\sqrt{1-b^2}} \frac{t^{q-1}}{(t^2 + ab^2)^{q/2}} dt db. \quad (2.10)$$

We have two cases, $q = 1$ or $q \geq 2$. Let us consider first the case $q = 1$ and $s = 1/2$. Equation (2.10) becomes

$$I = a^{1/2} \frac{2(n-1)c_{n-1}}{c_n} \int_0^1 b^{n-1} \int_0^{\sqrt{1-b^2}} \frac{1}{\sqrt{t^2 + ab^2}} dt db.$$

Integrating with respect to t we get

$$I = a^{1/2} \frac{2(n-1)c_{n-1}}{c_n} \int_0^1 b^{n-1} \ln\left(\frac{\sqrt{1-b^2} + \sqrt{1-(1-a)b^2}}{b\sqrt{a}}\right) db.$$

Since $a < 1$, we have $\sqrt{1-b^2} \leq \sqrt{1-(1-a)b^2}$. We upper bound I as follows

$$\begin{aligned} I &\leq a^{1/2} \frac{2(n-1)c_{n-1}}{c_n} \int_0^1 b^{n-1} \ln \left(\frac{2\sqrt{1-(1-a)b^2}}{b\sqrt{a}} \right) db \\ &\leq a^{1/2} \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \ln \left(\frac{1}{a} \right) + a \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \left(2 + \frac{2}{n} \right). \end{aligned}$$

This proves the case $q = 1$.

Finally, consider the case $q \geq 2$ and $s \geq 1$. From (2.10), since

$$(t^2 + ab^2)^{q/2} \geq t^q + (q/2)t^{2(q/2-1)}ab^2,$$

we have

$$\begin{aligned} I &\leq a^{q/2}\gamma \int_0^1 b^{n-1} \int_0^{\sqrt{1-b^2}} \frac{t^{q-1}}{t^q + (q/2)t^{2(q/2-1)}ab^2} dt db \\ &= a^{q/2}\gamma \int_0^1 b^{n-1} \int_0^{\sqrt{1-b^2}} \frac{t}{t^2 + (q/2)ab^2} dt db. \end{aligned}$$

where $\gamma = (qc_q(n-q)c_{n-q})/c_n$. Solving last integral, we get

$$\begin{aligned} I &\leq a^{q/2} \frac{\gamma}{2} \int_0^{\sqrt{1-b^2}} \frac{t^{q-1}}{t^q + (q/2)t^{2(q/2-1)}ab^2} dt db \\ &= a^{q/2}\gamma \int_0^1 b^{n-1} \int_0^{\sqrt{1-b^2}} \frac{t}{t^2 + (q/2)ab^2} dt db. \end{aligned}$$

where $\gamma = (qc_q(n-q)c_{n-q})/c_n$. Solving last integral, we get

$$\begin{aligned} I &\leq a^{q/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \ln \left(\frac{1 - (1 - q/2a)b^2}{q/2ab^2} \right) db \\ &= a^{q/2} \frac{\gamma}{2n} \ln \left(\frac{2}{qa} \right) + a^{q/2} \frac{\gamma}{n^2} + a^{q/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \ln \left(1 - (1 - q/2a)b^2 \right) db. \end{aligned} \quad (2.11)$$

If $a \leq 2/q$, then $\ln(1 - (1 - q/2a)b^2) \leq 0$. We upper bound previous equation as

$$I \leq a^{q/2} \frac{\Gamma(n/2)}{\Gamma(q/2)\Gamma((n-q)/2)} \ln \left(\frac{1}{a} \right) + a^{q/2} \frac{\Gamma(n/2)}{\Gamma(q/2)\Gamma((n-q)/2)} \frac{2}{n}.$$

If $a \geq 2/q$, then $\ln(1 - (1 - q/2a)b^2) \leq \ln(q/2)$. From (2.11), we have

$$\begin{aligned} I &\leq a^{q/2} \frac{\gamma}{2n} \ln \left(\frac{2}{qa} \right) + a^{q/2} \frac{\gamma}{n^2} + a^{q/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \ln \left(\frac{q}{2} \right) db \\ &= a^{q/2} \frac{\gamma}{2n} \ln \left(\frac{2}{qa} \right) + a^{q/2} \frac{\gamma}{n^2} \\ &= a^{q/2} \frac{\Gamma(n/2)}{\Gamma(q/2)\Gamma((n-q)/2)} \ln \left(\frac{1}{a} \right) + a^{q/2} \frac{\Gamma(n/2)}{\Gamma(q/2)\Gamma((n-q)/2)} \frac{2}{n}. \end{aligned}$$

The case $2s = q$ is completed by rewriting the above equation in terms of s .

We examine the case $2s > q$. Let $\|b\|_{n-q}^2 = \sum_{i=q}^n b_i^2$, and $\|t\|_q^2 = \sum_{i=1}^q b_i^2$, and let B'_q the q -dimensional ball of radius $\sqrt{1 - \|b\|_{n-q}^2}$. We have

$$\begin{aligned} I &= \frac{a^s}{c_n} \int_{B_{n-q}} \|b\|_{n-q}^{2s} \int_{B'_q} \frac{1}{(\|t\|_q^2 + a\|b\|_{n-q}^2)^s} dt db \\ &= \gamma \int_0^1 b^{n-q-1} \int_0^{\sqrt{1-b^2}} \frac{t^{q-1}}{(t^2/(ab^2) + 1)^s} dt db \end{aligned}$$

where $\gamma = q(n-q)c_q c_{n-q}/c_n$. Changing variables by setting $z = t^2/(ab^2)$, we get

$$\begin{aligned} I &= a^{q/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \int_0^{(1-b^2)/(ab^2)} \frac{z^{q/2-1}}{(z+1)^s} dz db, \\ &\leq a^{q/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \int_0^\infty \frac{z^{q/2-1}}{(z+1)^s} dz db = a^{q/2} \frac{\gamma}{2n} \int_0^\infty \frac{z^{q/2-1}}{(z+1)^s} dz. \end{aligned}$$

Applying formula [3.194, 3] of [GR94] and simplifying, we get

$$I = a^{q/2} \frac{\Gamma(n/2) \Gamma((2s-q)/2)}{\Gamma((n-q)/2) \Gamma(s)}.$$

□

We prove the following lower bound on integral (2.1).

Lemma 2.1.3 *Under the same hypothesis of lemma 2.1.1 we have*

$$I \geq \begin{cases} a^s \frac{\Gamma((q-2s)/2) \Gamma((n-q+2s)/2)}{\Gamma(q/2) \Gamma((n-q)/2)} - \gamma & \text{for } 2s < q \\ a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-2s)/2)} \ln\left(\frac{1}{a}\right) - \gamma' & \text{for } 2s = q \\ a^{q/2} \frac{\Gamma(n/2) \Gamma((2s-q)/2)}{\Gamma((n-q)/2) \Gamma(s)} - \gamma'' & \text{for } 2s > q, \end{cases} \quad (2.12)$$

where

$$\gamma = \begin{cases} a^{q/2} \frac{\Gamma((q-2s)/2) \Gamma((n+4s-q)/2) \Gamma(n/2)}{\Gamma(q/2) \Gamma(n/2+s) \Gamma((n-q)/2)} F\left(\frac{n}{2}, \frac{q-2s}{2}; \frac{n+2s}{2}; 1-a\right) & \text{if } s < 1 \\ a^{q/2} \frac{2s(n+2s-q)}{n} \frac{\Gamma((q-2s)/2) \Gamma((n+2s-q)/2)}{\Gamma(q/2) \Gamma((n-q)/2)} F\left(\frac{n}{2}, \frac{q-2s}{2}; \frac{n+2}{2}; 1-a\right), & \text{if } s \geq 1 \end{cases}$$

$$\gamma' = a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-q)/2)} \left[2n - 2n \ln 2 - \frac{2}{n} + \frac{s-1}{(n+2)(n+4)} F\left(1, \frac{n}{2}; \frac{n}{2} + 2; 1-a\right) \right], \quad (2.13)$$

$$\gamma'' = a^{q/2+1} \frac{q \Gamma(n/2+1) \Gamma((2s-q)/2)}{2s \Gamma(s) \Gamma((n-q)/2)} F\left(1, \frac{q}{2} + 1; s+1; 1-a\right).$$

Proof. Using formula [4.642] of [GR94], from (2.2) we get

$$I = \gamma \int_0^1 b^{n-q-1} \int_0^{\sqrt{1-b^2}} t^{q-1} \left(\frac{ab^2}{t^2 + ab^2} \right)^s dt db, \quad (2.14)$$

where $\gamma = q(n-q)c_q c_{n-q}/c_n$. Let $\alpha = ab^2$ and $f(\alpha)$ denote the second integral in equation (2.14).

$$f(\alpha) = \int_0^{\sqrt{1-b^2}} t^{q-1} \left(\frac{\alpha}{t^2 + \alpha} \right)^s dt. \quad (2.15)$$

First consider the case $2s < q$. We rewrite $f(\alpha)$ as

$$f(\alpha) = \alpha^s \left(\int_0^{\sqrt{1-b^2}} t^{q-2s-1} dt - \int_0^{\sqrt{1-b^2}} g(t) dt \right), \quad (2.16)$$

where

$$g(t) = t^{q-2s-1} \left(1 - \left(\frac{t^2}{t^2 + \alpha} \right)^s \right).$$

By setting $y = t^2/\alpha$, we have

$$\int_0^{\sqrt{1-b^2}} g(t) dt = \frac{\alpha^{(q-2s)/2}}{2} \int_0^{(1-b^2)/\alpha} y^{(q-2s)/2-1} \frac{(y+1)^s - y^s}{(y+1)^s} dy. \quad (2.17)$$

We consider two cases: $s < 1$ and $s \geq 1$. Let us start with $s < 1$. Notice that $(y+1)^s - y^s \leq 1$. Then from (2.17) we get

$$\begin{aligned} \int_0^{\sqrt{1-b^2}} g(t) dt &\leq \frac{\alpha^{(q-2s)/2}}{2} \int_0^{(1-b^2)/\alpha} \frac{y^{(q-2s)/2-1}}{(y+1)^s} dy \\ &= \frac{(1-b^2)^{(q-2s)/2}}{q-2s} F \left(s, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{\alpha} \right), \end{aligned}$$

due to formula [3.194, 1] of [GR94]. Substituting into (2.16) and solving the first integral, we have

$$f(\alpha) \geq \alpha^s \frac{(1-b^2)^{(q-2s)/2}}{q-2s} - \alpha^s \frac{(1-b^2)^{(q-2s)/2}}{q-2s} F \left(s, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{\alpha} \right).$$

Substituting ab^2 to α , (2.14) becomes

$$\begin{aligned} I &\geq a^s \frac{\gamma}{q-2s} \int_0^1 b^{n+2s-q-1} (1-b^2)^{(q-2s)/2} db - \\ &\quad - a^s \frac{\gamma}{q-2s} \int_0^1 b^{n+2s-q-1} (1-b^2)^{(q-2s)/2} F \left(s, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{ab^2} \right) db. \end{aligned}$$

By definition of the beta function (1.7) and by setting $z = (1-b^2)/(ab^2)$ in the second integral, we get

$$\begin{aligned} I &\geq a^s \frac{\gamma}{2(q-2s)} B \left(\frac{n+2s-q}{2}, \frac{q-2s}{2} + 1 \right) - \\ &\quad - \frac{\gamma}{2(q-2s)a^{(n-q)/2}} \int_0^{+\infty} \frac{z^{(q-2s)/2}}{(z+1/a)^{n/2+1}} F \left(s, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -z \right) dz. \end{aligned}$$

Applying formula [7.512, 10] of [GR94], substituting γ and simplifying, we have

$$\begin{aligned} I &\geq a^s \frac{\Gamma((n+2s-q)/2) \Gamma((q-2s)/2)}{\Gamma(q/2) \Gamma((n-q)/2)} - \\ &\quad - a^{-(n-q)/2} \frac{\Gamma((q-2s)/2) \Gamma((n+4s-q)/2) \Gamma(n/2)}{\Gamma(q/2) \Gamma((n+2s)/2) \Gamma((n-q)/2)} F \left(\frac{n+4s-q}{2}, \frac{n}{2}; \frac{n+2s}{2}; 1 - \frac{1}{a} \right). \end{aligned}$$

We conclude the case $s < 1$ by applying transformation formula [9.131, 1] of [GR94] to the hypergeometric function.

Let $s \geq 1$. By Lagrange's Theorem there exists a value ξ , $y \leq \xi \leq y+1$, such that $(y+1)^s - y^s = s \xi^{s-1}$. Since $\xi^{s-1} \leq (y+1)^{s-1}$, from equation (2.17) we get

$$\begin{aligned} \int_0^{\sqrt{1-b^2}} g(t) dt &\leq \alpha^{(q-2s)/2} \frac{s}{2} \int_0^{(1-b^2)/\alpha} \frac{y^{(q-2s)/2-1}}{y+1} dy \\ &= (1-b^2)^{(q-2s)/2} \frac{s}{q-2s} F \left(1, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{\alpha} \right) \end{aligned}$$

which follows from formula [3.194, 1] of [GR94]. Substituting into (2.16), we get

$$f(\alpha) \geq \alpha^s \frac{(1-b^2)^{(q-2s)/2}}{q-2s} - \alpha^s \frac{s(1-b^2)^{(q-2s)/2}}{q-2s} F\left(1, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{\alpha}\right).$$

Equation (2.14) becomes

$$I \geq a^s \frac{\gamma}{q-2s} \int_0^1 b^{n+2s-q-1} (1-b^2)^{(q-2s)/2} db - \alpha^s \frac{\gamma s}{q-2s} \int_0^1 b^{n+2s-q-1} (1-b^2)^{(q-2s)/2} F\left(1, \frac{q-2s}{2}; \frac{q-2s}{2} + 1; -\frac{1-b^2}{ab^2}\right) db.$$

Solving the first integral, setting $z = (1-b^2)/(ab^2)$ in the second integral and using formula [7.512, 10] of [GR94] we get

$$I \geq a^s \frac{\Gamma((n+2s-q)/2) \Gamma((q-2s)/2)}{\Gamma(q/2) \Gamma((n-q)/2)} - a^{q/2} \delta,$$

where

$$\delta = \frac{2s(n+2s-q) \Gamma((q-2s)/2) \Gamma((n+2s-q)/2)}{n \Gamma(q/2) \Gamma((n-q)/2)} F\left(\frac{n}{2}, \frac{q-2s}{2}; \frac{n}{2} + 1; 1-a\right).$$

This concludes the proof for $2s < q$.

Let $2s = q$. The function $f(\alpha)$ in (2.15) becomes

$$f(\alpha) = \int_0^{\sqrt{1-b^2}} t^{2s-1} \left(\frac{\alpha}{t^2 + \alpha}\right)^s dt.$$

We rewrite $f(\alpha)$ as

$$f(\alpha) = \alpha^s \left(\int_0^{\sqrt{1-b^2}} \frac{t}{t^2 + \alpha} dt - \int_0^{\sqrt{1-b^2}} g(t) dt \right), \quad (2.18)$$

where

$$g(t) = \frac{t}{t^2 + \alpha} - \frac{t^{2s-1}}{(t^2 + \alpha)^s}. \quad (2.19)$$

We analyze separately the case $s = 1/2$ and the case $s \geq 1$. Let us start with $s = 1/2$. Since $g(t) < 0$, we get

$$f(\alpha) \geq \alpha^{1/2} \int_0^{\sqrt{1-b^2}} \frac{t}{t^2 + \alpha} dt = \frac{\alpha^{1/2}}{2} \ln\left(\frac{1-b^2 + \alpha}{\alpha}\right).$$

Substituting into (2.14) we have

$$I \geq a^{1/2} \frac{\gamma}{2} \int_0^1 b^{n-1} \ln\left(\frac{1 - (1-a)b^2}{ab^2}\right) db$$

which yields

$$I \geq a^{1/2} \frac{\gamma}{2} \left(\int_0^1 b^{n-1} \ln(1 - (1-a)b^2) db + \int_0^1 b^{n-1} \ln\left(\frac{1}{ab^2}\right) db \right). \quad (2.20)$$

We note that $\ln(1 - (1 - 1/2 a)b^2) \geq \ln(1 - b^2)$. In order to bound the first integral in (2.20), we use the fact that $b^{n-1} \ln(1 - b^2) \geq \ln(1 - b^2)$ since $1 - b^2 \leq 1$. We get

$$\int_0^1 b^{n-1} \ln(1 - (1 - 1/2 a)b^2) db \geq \int_0^1 \ln(1 - b^2) db = 2 \ln 2 - 2.$$

Substituting into (2.20) we obtain

$$\begin{aligned} I &\geq a^{1/2} \frac{\gamma}{2} \left(2 \ln 2 - 2 + \frac{2}{n^2} + \frac{1}{n} \ln \left(\frac{1}{a} \right) \right) \\ &= a^{1/2} \frac{\Gamma(n/2)}{\Gamma(1/2) \Gamma((n-1)/2)} \ln \left(\frac{1}{a} \right) + a^{1/2} \frac{\Gamma(n/2)}{\Gamma(1/2) \Gamma((n-1)/2)} \left(2n \ln 2 - 2n + \frac{2}{n} \right), \end{aligned}$$

that concludes the proof of the case $s = 1/2$, $q = 1$. Consider now the case $q \geq 2$ and $s \geq 1$. By (2.19) it follows

$$g(t) \leq \frac{t(t^2 + \alpha)^{s-1} - t^{2s-1}}{(t^2 + \alpha)^s}.$$

Setting $y = t^2/\alpha$ and using Lagrange's theorem, we have

$$\begin{aligned} \int_0^{\sqrt{1-b^2}} g(t) dt &\leq \int_0^{(1-b^2)/\alpha} \frac{(y+1)^{s-1} - y^{s-1}}{2(y+1)^s} dy \\ &\leq \frac{s-1}{2} \int_0^{(1-b^2)/\alpha} \frac{1}{(y+1)^2} dy = \frac{s-1}{2} \frac{1-b^2}{1-b^2+\alpha}. \end{aligned}$$

Substituting into (2.18) we get

$$\begin{aligned} f(\alpha) &\geq \alpha^s \left(\int_0^{\sqrt{1-b^2}} \frac{t}{t^2 + \alpha} dt - \frac{s-1}{2} \frac{1-b^2}{1-b^2+\alpha} \right) \\ &= \frac{\alpha^s}{2} \ln \left(\frac{1-b^2+\alpha}{\alpha} \right) - \frac{\alpha^s (s-1)}{2} \frac{1-b^2}{1-b^2+\alpha}. \end{aligned}$$

Since $\alpha = ab^2$, substituting into (2.14) we have

$$I \geq a^s \frac{\gamma}{2} \int_0^1 b^{n-1} \ln \left(\frac{1-(1-a)b^2}{ab^2} \right) db - a^s \frac{(s-1)\gamma}{2} \int_0^1 b^{n-1} \frac{1-b^2}{1-(1-a)b^2} db. \quad (2.21)$$

Since $\ln(1-(1-a)b^2) \geq \ln(1-b^2)$, we get

$$I \geq a^s \frac{\gamma}{2} \int_0^1 \ln(1-b^2) + a^s \frac{\gamma}{2n} \ln \left(\frac{1}{a} \right) + a^s \frac{\gamma}{n^2} - a^2 \frac{\gamma (s-1)}{4} B \left(\frac{n}{2}, 2 \right) F \left(1, \frac{n}{2}; \frac{n}{2} + 2; 1-a \right),$$

where the last integral of (2.21) is solved by using formula [3.197, 3] of [GR94]. Note that $\int_0^1 \ln(1-b^2) = 2 - 2 \ln 2$. Simplifying we obtain

$$\begin{aligned} I &\geq a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-q)/2)} \ln \left(\frac{1}{a} \right) - \\ &\quad - a^s \frac{\Gamma(n/2)}{\Gamma(s) \Gamma((n-q)/2)} \left[2n - 2n \ln 2 - \frac{2}{n} + \frac{s-1}{(n+2)(n+4)} F \left(1, \frac{n}{2}; \frac{n}{2} + 2; 1-a \right) \right], \end{aligned}$$

which concludes the proof of the case $2s = q$.

Finally, consider the case $2s > q$. Setting $y = t^2/\alpha$, equation (2.15) becomes

$$f(\alpha) = \frac{\alpha^{q/2}}{2} \int_0^{(1-b^2)/\alpha} \frac{y^{q/2-1}}{(y+1)^s} dy.$$

It can be rewritten as

$$f(\alpha) = \frac{\alpha^{q/2}}{2} \left[\int_0^\infty \frac{y^{q/2-1}}{(y+1)^s} dy - \int_{(1-b^2)/\alpha}^\infty \frac{y^{q/2-1}}{(y+1)^s} dy \right]. \quad (2.22)$$

The two integrals in (2.22) can be solved by using formula [3.194, 3] and [3.194, 2] of [GR94] respectively. We have

$$f(\alpha) = \frac{\alpha^{q/2}}{2} B\left(\frac{q}{2}, \frac{2s-q}{2}\right) - \frac{\alpha^s}{(2s-q)(1-b^2)^{(2s-q)/2}} F\left(s, \frac{2s-q}{2}; \frac{2s-q}{2} + 1; -\frac{\alpha}{1-b^2}\right).$$

Substituting into (2.14) we get

$$\begin{aligned} I &\geq a^{q/2} \frac{\gamma}{2} B\left(\frac{q}{2}, \frac{2s-q}{2}\right) \int_0^1 b^{n-1} db - \\ &\quad - a^s \frac{\gamma}{2s-q} \int_0^1 \frac{b^{n+2s-q-1} (2s-q)^{2s-q/2}}{(1-b^2)} F\left(s, \frac{2s-q}{2}; \frac{2s-q}{2} + 1; -\frac{ab^2}{1-b^2}\right) db. \end{aligned}$$

Solving the first integral and changing variables in the second by setting $z = ab^2/(1-b^2)$, we get

$$I \geq a^{q/2} \frac{\gamma}{2n} B\left(\frac{q}{2}, \frac{2s-q}{2}\right) - a^{q/2+1} \frac{\gamma}{4s-2q} \int_0^{+\infty} \frac{z^{(n+2s-q)/2-1}}{(z+a)^{n/2+1}} F\left(s, \frac{2s-q}{2}; \frac{2s-q}{2} + 1; -z\right) dz. \quad (2.23)$$

Since

$$\frac{z^{(n+2s-q)/2-1}}{(z+a)^{n/2+1}} \leq \frac{z^{(2s-q)/2}}{(z+a)^2},$$

substituting into (2.23) and using [7.51, 10] of [GR94] we have

$$I \geq a^{q/2} \frac{\gamma}{2n} B\left(\frac{q}{2}, \frac{2s-q}{2}\right) - a^{q/2+1} \frac{\gamma \Gamma((2s-q)/2 + 1) \Gamma(q/2 + 1)}{(4s-2q) \Gamma(s+1)} F\left(1, \frac{q}{2} + 1; s+1; 1-a\right).$$

Substituting the constant γ we get

$$I \geq a^{q/2} \frac{\Gamma(n/2) \Gamma((2s-q)/2)}{\Gamma((n-q)/2) \Gamma(s)} - a^{q/2+1} \frac{q \Gamma(n/2 + 1) \Gamma((2s-q)/2)}{2s \Gamma(s) \Gamma((n-q)/2)} F\left(1, \frac{q}{2} + 1; s+1; 1-a\right).$$

This completes the proof of the lemma. \square

2.2 Eigenvalues Approximation by the Power Method

In this section we analyze the behavior of the randomized error of the power method for eigenvalue approximation, that is, we study the following n -dimensional integral

$$\begin{aligned} e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) &= \left(\int_{S_n} \left(\frac{\lambda_1 - \xi_k^{\text{Pow}}}{\lambda_1} \right)^p \mu(db) \right)^{1/p} \\ &= \left(\int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 \lambda_i^{2(k-1)} (1 - \lambda_i/\lambda_1)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 \lambda_i^{2(k-1)}} \right)^p \mu(db) \right)^{1/p} \\ &= \left(\int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2(k-1)} (1 - x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2(k-1)}} \right)^p \mu(db) \right)^{1/p}, \end{aligned} \quad (2.24)$$

where $x_i = \lambda_i/\lambda_1$, and $1 = x_1 = \dots = x_r > x_{r+1} \geq \dots \geq x_n > 0$.

The first part of this section is devoted to prove bounds independent of the particular matrix to which the algorithm is applied. In the second part, we give bounds that depend on the eigenvalues of the matrix.

2.2.1 Distribution free bounds

In [KW92a] a sharp bound for the power method is given for the \mathcal{L}_1 norm. These bounds do not depend on the distribution of eigenvalues and show that, for any matrix, the randomized error vanishes as the number of iteration grows. We report briefly these results.

Theorem 2.2.1 [KW92a] *Let ξ_k^{Pow} be the k -th value returned by the power method defined by (1.1). For any symmetric positive definite matrix A and for any $k \geq 2$, we have*

a)

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, 1) \leq \alpha(n) \frac{\ln n}{k-1},$$

where $\pi^{-1/2} \leq \alpha(n) \leq 0.871$ and for large n , $\alpha(n) \approx \pi^{-1/2}$. Moreover, for any $k > 1 + \frac{1}{2} \ln(n/\ln n)$, let A be a matrix with exactly two distinct eigenvalues $\lambda_1 > 0$ and $\lambda_i = \lambda_1(1 - \ln(n/\ln n)/(2(k-1)))$, for $i = 2, 3, \dots, n$. Then, for large n and k ,

b)

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, 1) \geq 0.5 \frac{\ln n}{k-1} (1 + o(1)).$$

Proof. See the proof of Theorem 3.1 in [KW92a]. □

This theorem states that no matter what the distribution of eigenvalues is nor how poorly separated the two largest eigenvalues are, the randomized error of the power method for eigenvalue approximation is bounded by a quantity that goes to zero as $\ln n/(k-1)$. Part b) states that this upper bound is sharp since for every k we can find a symmetric positive definite matrix that essentially achieves that bound. However, numerical experiments show that these estimates, precisely because they are general, are not always very accurate. Moreover, these bounds hold in the case the \mathcal{L}_1 norm is studied. The generalization to the \mathcal{L}_p case is not easy. In fact, the techniques used in [KW92a] for $p = 1$ cannot be used for an arbitrary p . The main difficulty is that we have to deal with hypergeometric functions instead of logarithms and arctangents.

We now assume to know the multiplicity r of the largest eigenvalue λ_1 . We show how this knowledge allows us very often to get tighter bounds than those in [KW92a]. This fits with the general paradigm that knowing more about the problem allows us to get better bounds. These bounds can be considered as independent of the distribution of the eigenvalues since they hold for all the matrices that have the largest eigenvalue with a given multiplicity.

Theorem 2.2.2 *Let ξ_k^{Pow} be the value returned by the power method after k steps as approximation of λ_1 . Let r , $r < n$, be the multiplicity of λ_1 . Let $C = (1 - 1/(2k-1))^{2(k-1)}$, $e^{-1} \leq C \leq 0.45$. For every*

$k \geq 2$ we have

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) \leq \begin{cases} C \frac{1}{2k-1} \left(\frac{\Gamma((r-2p)/2) \Gamma((n-r+2p)/2)}{\Gamma(r/2) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } 2p < r \\ C \frac{\ln^{1/p}(2k-1)}{2k-1} \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-2p)/2)} \right)^{1/p} + \beta & \text{for } 2p = r \\ C^{r/(2p)} \left(\frac{1}{2k-1} \right)^{r/(2p)} \left(\frac{\Gamma(n/2) \Gamma((2p-r)/2)}{\Gamma(p) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } 2p > r, \end{cases}$$

where

$$\beta = C \frac{1}{2k-1} \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-r)/2)} \right) \left(3 + \frac{2}{n} \right)^{1/p}.$$

Moreover, there exists a positive definite matrix with only two distinct eigenvalues for which we have

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) \geq \begin{cases} C \frac{1}{2k-1} \left(\frac{\Gamma((r-2p)/2) \Gamma((n-r+2p)/2)}{\Gamma(r/2) \Gamma((n-r)/2)} \right)^{1/p} - \frac{1}{2k-1} \gamma & \text{for } 2p < r \\ C \frac{1}{2k-1} \ln^{1/p} \left(\frac{1}{C} \right) \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-2p)/2)} \right)^{1/p} - \frac{1}{2k-1} \gamma' & \text{for } 2p = r \\ C^{r/(2p)} \frac{1}{2k-1} \left(\frac{\Gamma(n/2) \Gamma((2p-r)/2)}{\Gamma(p) \Gamma((n-r)/2)} \right)^{1/p} - \frac{1}{2k-1} \gamma'' & \text{for } 2p > r, \end{cases}$$

where $\gamma, \gamma', \gamma''$ are lower order terms (with respect to k) defined accordingly with lemma 2.1.3 (equation (2.13)).

Proof. By definition of the randomized error (2.24) we have

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) = \left(\int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2(k-1)} (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2(k-1)}} \right)^p db \right)^{1/p}. \quad (2.25)$$

Since $(1-x_i) \leq 1$, we have

$$\begin{aligned} [e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)]^p &\leq \int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2(k-1)} (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2(k-1)} (1-x_i)} \right)^p db \\ &= \int_{S_n} \left(1 - \frac{\sum_{i=1}^r b_i^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2(k-1)} (1-x_i)} \right)^p db. \end{aligned} \quad (2.26)$$

Note that

$$x_i^{2(k-1)} (1-x_i) \leq \max_{x \in (0,1)} x^{2(k-1)} (1-x) = \left(1 - \frac{1}{2k-1} \right)^{2(k-1)} \frac{1}{2k-1}.$$

Then we upper bound equation (2.26) as follows

$$\begin{aligned} [e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)]^p &\leq \int_{S_n} \left(1 - \frac{\sum_{i=1}^r b_i^2}{\sum_{i=1}^r b_i^2 + a \sum_{i=r+1}^n b_i^2} \right)^p db \\ &= \int_{S_n} \left(\frac{a \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + a \sum_{i=r+1}^n b_i^2} \right)^p db, \end{aligned}$$

where $a = (1 - 1/(2k - 1))^{2(k-1)} / (2k - 1)$. The theorem is proven by observing that $0 < a < 1$ and applying the bounds proved in lemma 2.1.2 (see formula (2.7)).

We now prove the existence of a matrix for which the randomized error is lower bounded by the lower bound in Theorem 2.2.2. Let A be a $n \times n$ matrix whose eigenvalues are as follows. A has only two distinct eigenvalues, λ_1 with multiplicity r and λ_{r+1} with multiplicity $n - r$. For this matrix we get

$$[e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)]^p = \frac{1}{c_n} (1 - x_{r+1})^p \int_{B_n} \left(\frac{x_{r+1}^{2(k-1)} \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2(k-1)} \sum_{i=r+1}^n b_i^2} \right)^p db. \quad (2.27)$$

We can apply to the integral (2.27) the lower bounds proved in lemma 2.1.3. We get

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) \geq \begin{cases} (1 - x_{r+1}) \left[x_{r+1}^{2(k-1)} \left(\frac{\Gamma(n + 2p - r)/2 \Gamma((r - 2p)/2)}{\Gamma(r/2) \Gamma((n - r)/2)} \right)^{1/p} - \gamma \right] & 2p < r \\ (1 - x_{r+1}) \left[x_{r+1}^{2(k-1)} \ln^{1/p} \left(\frac{1}{x_{r+1}^{2(k-1)}} \right) \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n - r)/2)} \right)^{1/p} - \gamma' \right] & 2p = r \\ (1 - x_{r+1}) \left[x_{r+1}^{r(k-1)/p} \left(\frac{\Gamma(n/2) \Gamma((2p - r)/2)}{\Gamma(p) \Gamma((n - r)/2)} \right)^{1/p} - \gamma'' \right] & 2p > r, \end{cases}$$

where γ, γ' and γ'' are as in lemma 2.1.3. By setting $x_{r+1} = 1 - 1/(2k - 1)$ we conclude the proof. \square

Note that for $2p < r$ the upper bound is sharp. For the other cases, $2p = r$ and $2p > r$, we have not been able to produce a matrix which achieves the upper bound. For $p = 1$ it is possible to compare these bounds with those reported in Theorem 2.2.1. We have that for small r the bounds in 2.2.1 are better, while for $r \geq n/\ln n$ our bounds are tighter. We have anyway to underline that these bounds estimate the randomized error in the \mathcal{L}_p case, while Kuczyński and Woźniakowski dealt with the easier \mathcal{L}_∞ case.

2.2.2 Distribution Dependent Bounds

In this section we give upper and lower bounds on the randomized error for eigenvalue approximation with the power method. In [KW92a] the randomized error for the power method has been studied considering the \mathcal{L}_1 norm. In that paper the authors study the rate of convergence as the number of iterations k goes to infinity. The following theorems establish upper and lower bounds on the randomized error for all $k \geq 1$ and for every $p \in [1, \infty)$.

Theorem 2.2.3 *Let A be a symmetric positive definite matrix, and let $r < n$ be the multiplicity of the*

largest eigenvalue λ_1 of A . For every p , $1 \leq p < \infty$ and for every $k \geq 1$, we have

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) \leq \begin{cases} x_{r+1}^{2k} (1 - x_n) \left(\frac{\Gamma((r-2p)/2) \Gamma((n-r+2p)/2)}{\Gamma((n-r)/2) \Gamma(r/2)} \right)^{1/p} & \text{if } 2p < r, \\ k^{1/p} x_{r+1}^{2k} (1 - x_n) \left(\ln \left(\frac{1}{x_{r+1}} \right) \frac{2\Gamma(n/2)}{\Gamma((n-2p)/2) \Gamma(p)} \right)^{1/p} + \beta & \text{if } 2p = r, \\ x_{r+1}^{kr/p} (1 - x_n) \left(\frac{\Gamma((2p-r)/2) \Gamma(n/2)}{\Gamma(p) \Gamma((n-r)/2)} \right)^{1/p} & \text{if } 2p > r \end{cases}$$

where $x_i = \lambda_i/\lambda_1$, and

$$\beta = x_{r+1}^{2k} \left(1 - \frac{\lambda_n}{\lambda_1} \right) \left(\frac{\Gamma(n/2)}{\Gamma((n-2p)/2) \Gamma(p)} \left(2 + \frac{2}{n} \right) \right)^{1/p}.$$

Proof. By equation (2.24) we get

$$\begin{aligned} [e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)]^p &= \int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k} (1 - x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^p \mu(db) \\ &\leq (1 - x_n)^p \int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^p \mu(db) \\ &\leq (1 - x_n)^p \int_{S_n} \left(\frac{x_{r+1}^{2k} \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} \sum_{i=r+1}^n b_i^2} \right)^p \mu(db). \end{aligned}$$

The theorem now follows by using the upper bounds (2.7) in lemma 2.1.2. \square

Using the same arguments, and the lower bounds (2.12) in lemma 2.1.3, we get the following result.

Theorem 2.2.4 *Let A be a symmetric positive definite matrix, and let $r < n$ be the multiplicity of the largest eigenvalue λ_1 of A . Then, for every p , $1 \leq p < \infty$ and for every $k \geq 1$, we have*

$$e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) \geq \begin{cases} x_{r+1}^{2k} (1 - x_{r+1}) \left(\frac{\Gamma((r-2p)/2) \Gamma((2p+1)/2)}{\Gamma(1/2) \Gamma(r/2)} \right)^{1/p} - x_{r+1}^{kr/p} \gamma & \text{for } 2p < r, \\ k^{1/p} x_{r+1}^{2k} (1 - x_{r+1}) \left(\ln \left(\frac{1}{x_{r+1}} \right) \frac{2\Gamma((2p+1)/2)}{\Gamma(1/2) \Gamma(p)} \right)^{1/p} - x_{r+1}^{2k} \gamma' & \text{for } 2p = r, \\ x_{r+1}^{kr/p} (1 - x_{r+1}) \left(\frac{\Gamma((2p-r)/2) \Gamma((r+1)/2)}{\Gamma(p) \Gamma(1/2)} \right)^{1/p} - x_{r+1}^{(kr+2)/p} \gamma'' & \text{for } 2p > r \end{cases}$$

where $x_{r+1} = \lambda_{r+1}/\lambda_1$ and $\gamma, \gamma', \gamma''$ are lower order terms (with respect to k) defined accordingly to lemma 2.1.3 (equation (2.13)) as follows

$$\begin{aligned} \gamma &= (1 - x_{r+1}) \left(\frac{p(2p+1) \Gamma((r-2p)/2) \Gamma((2p+1)/2)}{r+1 \Gamma(r/2) \Gamma(1/2)} F \left(\frac{r+1}{2}, \frac{r-2p+2}{2}; \frac{r+3}{2}; 1 - x_{r+1}^{2k} \right) \right)^{1/p}, \\ \gamma' &= (1 - x_{r+1}) \left(\frac{\Gamma((2p+1)/2)}{\Gamma(p) \Gamma(1/2)} \left(\frac{2}{2p+1} - \frac{2(p+1)}{2p+3} + (\ln 2 - 1)(4p+2) + \ln 2 \right) \right)^{1/p}, \\ \gamma'' &= (1 - x_{r+1}) \left(\frac{\Gamma((r+3)/2) \Gamma((2p-r)/2)}{2p\Gamma(p) \Gamma(1/2)} F \left(1, \frac{r}{2} + 1; p+1; 1 - x_{r+1}^{2k} \right) \right)^{1/p}. \end{aligned}$$

Proof. By (2.24) we have

$$\begin{aligned}
[e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)]^p &= \int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k} (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \mu(db) \right)^p \\
&\geq (1-x_{r+1})^p \int_{\|b\|=1} \left(\frac{x_{r+1}^{2k} b_{r+1}^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} b_{r+1}^2} \right)^p \mu(db) \\
&\geq (1-x_{r+1})^p \int_{\|b\|_{r+1}=1} \left(\frac{x_{r+1}^{2k} b_{r+1}^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} b_{r+1}^2} \right)^p \mu(db),
\end{aligned}$$

where $\|b\|_{r+1}^2 = \sum_{i=1}^{r+1} b_i^2$. We then apply the lower bound of lemma 2.1.3 remembering that the above integral is over the unit sphere of dimension $r+1$. \square

We note that the upper and lower bounds of the previous two theorems have the same behavior (up to a multiplicative constant) with respect to k . Using similar techniques, we can derive the formulas for the asymptotic rate of convergence for all $p \in [1, +\infty)$. These formulas show that the constants in the previous theorems are asymptotically optimal. The existence of tight upper and lower bounds shows that for the power method the speed of convergence of the randomized error does depend on the relation between r and p . In other words, the existence of three different cases is intrinsic for the problem. A similar phenomenon holds also for the randomized error for eigenvector estimate with the power method as we will see in the following section.

We already comment on the fact that we were not able to find sharp distribution free bounds for the general \mathcal{L}_p case. The behavior of the power method underlined by Theorems 2.2.3 and 2.2.4 gives us the intuition that, probably, we have the same dependence on p and r also for the bounds that do not depend on λ_{r+1}/λ_1 . In fact, we believe that, in order to get matching upper and lower bounds, we have to make some effort to improve the lower bound rather than the upper bound. In fact, from Theorem 2.2.2 we have that the lower bound goes to zero as $1/(2k+1)$ independently of the relation between p and r .

2.3 Eigenvector approximation by the Power Method

We now analyze the randomized error of the power method for eigenvector estimate. The randomized error in the sense of \mathcal{L}_p , $1 \leq p < +\infty$, for eigenvector approximation by the power method can be defined as follows

$$\begin{aligned}
e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) &= \left(\int_{S_n} \inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\text{Pow}} - \mathbf{v}\|^p \mu(db) \right)^{1/p} \\
&= \left(\int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^{p/2} \mu(db) \right)^{1/p}, \tag{2.28}
\end{aligned}$$

where $x_i = \lambda_i/\lambda_1$. We note that the minimum distance between the vector $\mathbf{u}_k^{\text{Pow}}$ and the eigenspace \mathbf{Z}_1 corresponds to the sine of the angle $\alpha_k(\mathbf{b})$ between the vector $\mathbf{u}_k^{\text{Pow}}$ and its orthogonal projection on the subspace \mathbf{Z}_1 . In the same way we define the randomized error in the \mathcal{L}_∞ space to be

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, \infty) = \sup_{\mathbf{b} \in S_n} \inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\text{Pow}} - \mathbf{v}\| = \sup_{\mathbf{b} \in S_n} |\sin(\alpha_k(\mathbf{b}))|$$

$$= \sup_{\|\mathbf{b}\|=1} \sqrt{\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}}}. \quad (2.29)$$

It is easy to see that the supremum in (2.29) is achieved by setting $\sum_{i=1}^r b_i^2 = 0$. From (2.29), we get

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, \infty) = 1. \quad (2.30)$$

We first show that, on the contrary of what happens for eigenvalue, it is impossible to find sharp bounds that are independent of the matrix to which the method is applied. Then, we give sharp upper and lower bounds that essentially depend on the ratio between the two largest eigenvalue.

2.3.1 Worst Case Matrices

As we pointed out in Section 2.2.1, the randomized error for eigenvalue estimate by the power method can be bounded by a quantity that goes to zero as fast as $\ln(n)/k$ independently of the distribution of the eigenvalues. Our first goal is to analyze the possibility of obtaining distribution-free bounds for the problem of approximating a largest eigenvector. To this extent, we will deal with “worst case matrices”.

Let us denote by $s(k, p)$ the supremum of the randomized error in the \mathcal{L}_p sense over all positive definite matrices A , i.e.,

$$s(k, p) = \sup_{A=A^*>0} e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p).$$

Since the randomized error increases with x_i , see (2.28), it is easy to show that the supremum is achieved by setting $x_i = 1$ for every $i \geq 2$ and for every p , $1 \leq p < \infty$. Then we get

$$s(k, p) = \left[\int_{S_n} \left(\frac{\sum_{i=2}^n b_i^2}{b_1^2 + \sum_{i=2}^n b_i^2} \right)^{p/2} db \right]^{1/p} = \left[\int_{S_n} \left(1 - \frac{b_1^2}{\sum_{i=1}^n b_i^2} \right)^{p/2} db \right]^{1/p}. \quad (2.31)$$

Hence, $s(k, p)$ is independent of k and cannot go to zero. This shows that there are no distribution-free bounds. In fact, $s(k, p)$ are pretty close to 1. We first consider the case $p = 1$. Using (2.31) and symmetry argument, we have

$$s(k, 1) = \int_{S_n} \left(1 - \frac{b_1^2}{\sum_{i=1}^n b_i^2} \right)^{1/2} db \geq \int_{S_n} \left(1 - \frac{b_1^2}{\sum_{i=1}^n b_i^2} \right) db = \left(1 - \frac{1}{n} \right). \quad (2.32)$$

We obtain estimates on $s(k, p)$ by the following proposition.

Proposition 2.3.1 *For every k and p , $1 \leq p < \infty$, we have*

$$\left(1 - \frac{1}{n} \right) \leq s(k, p) \leq 1.$$

Proof. The right-hand side inequality is trivial. Let us prove the left-hand side. For $p = 1$ it follows immediately by (2.32). For $p > 1$, applying Hölder’s inequality to (2.31) we get

$$\int_{S_n} \left(1 - \frac{b_1^2}{\sum_{i=1}^n b_i^2} \right)^{1/2} db \leq \left[\int_{S_n} \left(1 - \frac{b_1^2}{\sum_{i=1}^n b_i^2} \right)^{p/2} db \right]^{1/p} \left[\int_{S_n} db \right]^{1/q},$$

where p and q are the conjugate exponents, i.e., $1/p + 1/q = 1$. The proof is completed by observing that $\int_{S_n} db = 1$. \square

Proposition 2.3.1 states that for every k there are matrices for which the randomized error is close to one. These matrices have the largest eigenvalue of multiplicity one, and the second largest eigenvalue has multiplicity $n-1$ and is pathologically close to λ_1 . In this case, even if the starting vector is random, the sequence $\{\mathbf{u}_i^{\text{Pow}}\}$ for $i = 1, 2, \dots, k$ does not approximate a largest eigenvector.

2.4 Non Asymptotic Behavior

So far we have seen that if $\lambda_{r+1}/\lambda_1 \approx 1$ then the power method behaves badly even for a random starting vector. We now analyze the relationship between the ratio λ_{r+1}/λ_1 and the rate of convergence of the power method for approximating a largest eigenvector. We first show upper and lower bounds on the randomized error $e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$. These bounds depend on the distribution of the eigenvalues of the matrix A and on the norm used. In particular, we prove that the rate of convergence is slower when the multiplicity of λ_1 is smaller than the value of the parameter p of the norm. What seems interesting about these results is that they hold for a complete class of norms, and we are able to show how the speed of convergence of the power method depends on the norm.

2.4.1 Upper Bounds

We now show how the rate of convergence depends on the multiplicity r of the largest eigenvalue and on the value of the parameter p of the norm. Theorem 2.4.1 shows that the rate of convergence depends on the relation between the parameters r and p . In particular the speed of convergence increases with r and decreases with p .

Theorem 2.4.1 *Let A be a symmetric positive definite matrix, and let $r, r < n$, denote the multiplicity of the largest eigenvalue λ_1 of A . Let*

$$\beta = \left[\frac{\Gamma(n/2)}{\Gamma(p/2)\Gamma((n-p)/2)} \left(2 + \frac{2}{n}\right) \right]^{1/p}.$$

Then, for every $p, 1 \leq p < \infty$, and for every k we have

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) \leq \begin{cases} x_{r+1}^k \left(\frac{\Gamma((r-p)/2)\Gamma((n+p-r)/2)}{\Gamma(r/2)\Gamma((n-r)/2)} \right)^{1/p} & \text{for } p < r, \\ x_{r+1}^k (2k)^{1/p} \left[\ln \left(\frac{1}{x_{r+1}} \right) \right]^{1/p} \left(\frac{\Gamma(n/2)}{\Gamma(p/2)\Gamma((n-p)/2)} \right)^{1/p} + \beta x_{r+1}^k & \text{for } p = r, \\ x_{r+1}^{kr/p} \left(\frac{\Gamma((p-r)/2)\Gamma(n/2)}{\Gamma(p/2)\Gamma((n-r)/2)} \right)^{1/p} & \text{for } p > r, \end{cases}$$

where $x_{r+1} = \lambda_{r+1}/\lambda_1$.

Proof. By equation (2.28) we have

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p = \int_{S_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^{p/2} db.$$

Observe that the integrand is an increasing function of $\sum_{i=r+1}^n b_i^2 x_i^{2k}$. The upper bound is then obtained by replacing x_i by x_{r+1} for $i > r + 1$,

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p \leq \int_{B_n} \left(\frac{x_{r+1}^{kp} \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} \sum_{i=r+1}^n b_i^2} \right)^{p/2} db. \quad (2.33)$$

The theorem is proven by applying the same reasoning as in the proof of Lemma 2.1.2. \square

Note that, when $p = r$, the bound is composed of two terms. The first term depends on k through $x_{r+1}^k k^{1/p}$, the second term depends on k through x_{r+1}^k . We remark that for large k the influence of the second term is negligible. Nevertheless, numerical tests show that this term can affect the bound when the value of x_{r+1} is close to 1.

2.4.2 Lower Bounds

In this section we find lower bounds on the randomized error $e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$. As in Section 2.4.1, we show that these lower bounds depend on the multiplicity of the largest eigenvalue and on the value of the parameter p of the norm. Upper and lower bounds show the same dependence on the ratio between the two largest eigenvalues and on the relation between p and r .

Below we define some constants that are used in Theorem 2.4.2.

$$\gamma = \begin{cases} \left(\frac{\Gamma((r-p)/2) \Gamma(p+1/2) \Gamma((r+1)/2)}{2\Gamma(r/2) \Gamma(1/2) \Gamma((r+p+1)/2)} F\left(\frac{r+1}{2}, \frac{r-p}{2}; \frac{r+p+1}{2}; 1-x_{r+1}^{2k}\right) \right)^{1/p}, & \text{for } p < 2, \\ \left(\frac{p\Gamma((r-p)/2) \Gamma((p+3)/2)}{2(r+1)\Gamma(r/2) \Gamma(1/2)} F\left(\frac{r+1}{2}, \frac{r-p}{2}; \frac{r+3}{2}; 1-x_{r+1}^{2k}\right) \right)^{1/p}, & \text{for } p \geq 2. \end{cases}$$

$$\gamma' = \left(\frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \left(\log\left(\frac{p^2+3p}{4}\right) - \frac{2}{p+1} + \frac{2p-4}{p+3} F\left(1, \frac{p+1}{2}; \frac{p+5}{2}; 1-x_{r+1}^{2k}\right) \right) \right)^{1/p},$$

$$\gamma'' = \left(\frac{r\Gamma((r+1)/2+1) \Gamma((p-r)/2)}{4p\Gamma(1/2) \Gamma(p/2+1)} F\left(\frac{r}{2}+1, 1; \frac{p}{2}+1; 1-x_{r+1}^{2k}\right) \right)^{1/p}.$$

Theorem 2.4.2 *Let A be a symmetric positive definite matrix, and let $r, r < n$, denote the multiplicity of the largest eigenvalue λ_1 of A . Then, for every $p, 1 \leq p < \infty$, and for every k we have*

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) \geq \begin{cases} x_{r+1}^k \left(\frac{\Gamma((r-p)/2) \Gamma((p+1)/2)}{\Gamma(r/2) \Gamma(1/2)} \right)^{1/p} - \gamma x_{r+1}^{kr/p} & \text{for } p < r, \\ x_{r+1}^k (2k)^{1/r} \left[\ln\left(\frac{1}{x_{r+1}}\right) \right]^{1/r} \left(\frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \right)^{1/p} - \gamma' x_{r+1}^k & \text{for } p = r, \\ x_{r+1}^{kr/p} \left(\frac{\Gamma((p-r)/2) \Gamma((r+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \right)^{1/p} - \gamma'' x_{r+1}^{k(r+2)/p} & \text{for } p > r, \end{cases}$$

where $x_{r+1} = \lambda_{r+1}/\lambda_1$.

Proof. From equation (2.28) and (1.9) we have

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p = \frac{1}{c_n} \int_{B_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^{p/2} db.$$

Notice that the integrand is an increasing function of $\sum_{i=r+1}^n b_i^2 x_i^{2k}$. Hence, the lower bound is obtained by replacing x_i by 0 for $i > r + 1$,

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p \geq \frac{x_{r+1}^{kp}}{c_n} \int_{B_n} \frac{b_{r+1}^p}{(\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} b_{r+1}^2)^{p/2}} db.$$

Let $a = x_{r+1}^k$. Writing the last integral as an integral over the ball B_{n-r} and the r dimensional ball of radius $q = \sqrt{1 - \sum_{i=r+1}^n b_i^2}$, and applying [4.642] of [GR94], we get

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p \geq \frac{r c_r}{c_n} \int_{B_{n-r}} \int_0^q t^{r-1} \left(\frac{a^2 b_{r+1}^2}{t^2 + a^2 b_{r+1}^2} \right)^{p/2} dt db. \quad (2.34)$$

Let us denote $a^2 b_{r+1}^2$ by α , and consider the integral

$$f(\alpha) = \int_0^q t^{r-1} \left(\frac{\alpha}{t^2 + \alpha} \right)^{p/2} dt. \quad (2.35)$$

We have three cases depending on the relation between p and r .

Consider first the case $p < r$. It is convenient to split $f(\alpha)$ as follows

$$f(\alpha) = \alpha^{p/2} \left(\int_0^q t^{r-p-1} dt - \int_0^q g(t) dt \right), \quad (2.36)$$

where

$$g(t) = t^{r-1} \left(\frac{1}{t^p} - \left(\frac{1}{t^2 + \alpha} \right)^{p/2} \right).$$

We can conveniently rewrite $g(t)$ as

$$g(t) = t^{r-p-1} \left(1 - \left(\frac{t^2}{t^2 + \alpha} \right)^{p/2} \right).$$

Setting $y = t^2/\alpha$, we have

$$\int_0^q g(t) dt = \frac{\alpha^{(r-p)/2}}{2} \int_0^{q^2/\alpha} y^{(r-p)/2-1} \frac{(y+1)^{p/2} - y^{p/2}}{(y+1)^{p/2}} dy. \quad (2.37)$$

We consider two cases: $p < 2$ and $p \geq 2$. Let us start with $p < 2$. Notice that $(y+1)^{p/2} - y^{p/2} \leq 1$. Then from (2.37) we get

$$\begin{aligned} \int_0^q g(t) dt &\leq \frac{\alpha^{(r-p)/2}}{2} \int_0^{q^2/\alpha} \frac{y^{(r-p)/2-1}}{(y+1)^{p/2}} dy \\ &= \frac{q^{r-p}}{r-p} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{q^2}{\alpha}\right), \end{aligned}$$

due to formula [3.194, 1] of [GR94] (see also [GR94] for the definition and the properties of the hypergeometric function $F(a, b; c; x)$). Substituting it into (2.36) and solving the first integral, we have

$$f(\alpha) \geq \alpha^{p/2} \frac{q^{r-p}}{r-p} - \alpha^{p/2} \frac{q^{r-p}}{r-p} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{q^2}{\alpha}\right).$$

Hence, (2.34) becomes

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq \frac{rc_r}{(r-p)c_n} \int_{B_{n-r}} a^p b_{r+1}^p \left(1 - \sum_{i=r+1}^n b_i^2\right)^{(r-p)/2} db - \\ &- \frac{rc_r}{(r-p)c_n} \int_{B_{n-r}} a^p b_{r+1}^p \left(1 - \sum_{i=r+1}^n b_i^2\right)^{(r-p)/2} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{1 - \sum_{i=r+1}^n b_i^2}{a^2 b_{r+1}^2}\right) db. \end{aligned}$$

Using [4.642] of [GR94], we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \frac{\Gamma((p+1)/2) \Gamma((r-p)/2)}{\Gamma(1/2) \Gamma(r/2)} - \\ &- a^p \frac{(r+1) \Gamma((r+1)/2)}{(r-p) \Gamma(r/2) \Gamma(1/2)} \int_0^1 t^p (1-t^2)^{(r-p)/2} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{1-t^2}{a^2 t^2}\right) dt. \end{aligned} \quad (2.38)$$

After setting $y = (1-t^2)/(a^2 t^2)$, we can rewrite the integral in (2.38) as

$$\begin{aligned} \int_0^1 t^p (1-t^2)^{(r-p)/2} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{1-t^2}{a^2 t^2}\right) dt &= \\ &= \frac{a^{-p-1}}{2} \int_0^\infty \frac{y^{(r-p)/2}}{(y+1/a^2)^{(r+3)/2}} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -y\right) dy. \end{aligned}$$

From the last equation and using formula [7.512, 10] of [GR94], we have

$$\begin{aligned} \int_0^1 t^p (1-t^2)^{(r-p)/2} F\left(\frac{p}{2}, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{1-t^2}{a^2 t^2}\right) dt &= \\ &= \frac{a^{-p-1}}{2} \frac{\Gamma((r-p)/2 + 1) \Gamma(p+1/2) \Gamma((r+1)/2)}{\Gamma((r+3)/2) \Gamma((p+r+1)/2)} F\left(p + \frac{1}{2}, \frac{r+1}{2}; \frac{p+r+1}{2}; 1 - \frac{1}{a^2}\right). \end{aligned} \quad (2.39)$$

Applying transformation formula to the hypergeometric function, see [9.131, 1] of [GR94], we have

$$F\left(p + \frac{1}{2}, \frac{r+1}{2}; \frac{p+r+1}{2}; 1 - \frac{1}{a^2}\right) = a^{r+1} F\left(\frac{r+1}{2}, \frac{r-p}{2}; \frac{p+r+1}{2}; 1 - a^2\right).$$

Substituting it into (2.39) and then into (2.38), we get

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p \geq a^p \frac{\Gamma((p+1)/2) \Gamma((r-p)/2)}{\Gamma(1/2) \Gamma(r/2)} - a^r \gamma,$$

where

$$\gamma = \frac{\Gamma((r-p)/2) \Gamma(p+1/2) \Gamma((r+1)/2)}{2\Gamma(r/2) \Gamma(1/2) \Gamma((r+p+1)/2)} F\left(\frac{r+1}{2}, \frac{r-p}{2}; \frac{p+r+1}{2}; 1 - a^2\right).$$

This concludes the proof of the case $p < 2$.

Let $p \geq 2$. Observe that, from Lagrange's Theorem, there exists a value ξ , $y \leq \xi \leq y+1$, such that $(y+1)^{p/2} - y^{p/2} = p/2 \xi^{p/2-1}$. Since $\xi^{p/2-1} \leq (y+1)^{p/2-1}$, we obtain the bound

$$\begin{aligned} \int_0^q g(t) dt &\leq \frac{\alpha^{(r-p)/2} p}{4} \int_0^{q^2/\alpha} \frac{y^{(r-p)/2-1}}{y+1} dy \\ &= q^{r-p} \frac{p}{2(r-p)} F\left(1, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{q^2}{\alpha}\right) \end{aligned} \quad (2.40)$$

which follows from formula [3.194, 1] of [GR94]. Proceeding exactly as before, we get

$$f(\alpha) \geq \alpha^{p/2} \frac{q^{r-p}}{r-p} - \alpha^{p/2} \frac{q^{r-p} p}{2(r-p)} F\left(1, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{q^2}{\alpha}\right).$$

Using this bound in (2.34), we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq \frac{r c_r}{(r-p) c_n} \int_{B_{n-r}} a^p b_{r+1}^p \left(1 - \sum_{i=r+1}^n b_i^2\right)^{(r-p)/2} db - \\ &- \frac{r p c_r}{2(r-p) c_n} \int_{B_{n-r}} a^p b_{r+1}^p \left(1 - \sum_{i=r+1}^n b_i^2\right)^{(r-p)/2} F\left(1, \frac{r-p}{2}; \frac{r-p}{2} + 1; -\frac{1 - \sum_{i=r+1}^n b_i^2}{a^2 b_{r+1}^2}\right) db. \end{aligned} \quad (2.41)$$

Solving the integral in (2.41) as before, and applying the transformation formula [9.131] of [GR94] to the hypergeometric function, we have

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p \geq a^p \frac{\Gamma((p+1)/2) \Gamma((r-p)/2)}{\Gamma(1/2) \Gamma(r/2)} - a^r \gamma,$$

where

$$\gamma = \frac{p \Gamma((r-p)/2) \Gamma((p+3)/2)}{2(r+1) \Gamma(r/2) \Gamma(1/2)} F\left(\frac{r+1}{2}, \frac{r-p}{2}; \frac{r+1}{2} + 1, 1 - a^2\right).$$

This concludes the proof for $p < r$.

Let $p = r$. The integral denoted by $f(\alpha)$ in (2.35) becomes

$$f(\alpha) = \alpha^{p/2} \int_0^q t^{p-1} \left(\frac{1}{t^2 + \alpha}\right)^{p/2} dt, \quad (2.42)$$

and can be rewritten as

$$f(\alpha) = \alpha^{p/2} \left(\int_0^q \frac{t}{t^2 + p/2 \alpha} dt - \int_0^q g(t) dt \right), \quad (2.43)$$

where

$$g(t) = \frac{t}{t^2 + p/2 \alpha} - \frac{t^{p-1}}{(t^2 + \alpha)^{p/2}}.$$

Since $p = r$, we have that p is an integer between 1 and n . We analyze separately the cases $p = 1$ and $p \geq 2$. If $p = 1$, then $g(t) \leq 0$ and

$$\begin{aligned} f(\alpha) &\geq \alpha^{1/2} \left(\int_0^q \frac{t}{t^2 + 1/2 \alpha} dt \right) \\ &= \frac{\alpha^{1/2}}{2} \ln \left(\frac{q^2 + 1/2 \alpha}{1/2 \alpha} \right). \end{aligned}$$

From (2.34) and since $q = \sqrt{1 - \sum_{i=2}^n b_i^2}$, we get

$$[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, 1)]^1 \geq \frac{1}{c_n} \int_{B_{n-1}} \alpha^{1/2} \ln \left(\frac{1 - \sum_{i=2}^n b_i^2 + 1/2 \alpha}{1/2 \alpha} \right) db.$$

Let $\|b\| = \sum_{i=3}^n b_i^2$, and $t = b_2/(1 - \|b\|^2)^{1/2}$. Since $\alpha = a^2 b_2^2$, using [4.642] of [GR94], we have

$$\begin{aligned}
[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, 1)] &\geq a \frac{2}{c_n} \int_{B_{n-2}} \int_0^1 t (1 - \|b\|^2) \ln \left(\frac{1 - (1 - 1/2 a^2) t^2}{1/2 a^2 t^2} \right) dt db \\
&= a \frac{(n-2)c_{n-2}}{c_n} B\left(\frac{n}{2} - 1, 2\right) \int_0^1 t \ln \left(\frac{1 - (1 - 1/2 a^2) t^2}{1/2 a^2 t^2} \right) dt \\
&= a \frac{(n-2)c_{n-2}}{c_n} B\left(\frac{n}{2} - 1, 2\right) \frac{1}{2(1 - 1/2 a^2)} \ln \left(\frac{2}{a^2} \right) \\
&\geq a \frac{(n-2)c_{n-2}}{2c_n} B\left(\frac{n}{2} - 1, 2\right) \ln \left(\frac{1}{a^2} \right) + a \frac{(n-2)c_{n-2}}{2c_n} B\left(\frac{n}{2} - 1, 2\right) \ln(2),
\end{aligned}$$

from which we have

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, 1) \geq \frac{a}{\pi} \ln \left(\frac{1}{a^2} \right) + \frac{a}{\pi} \ln(2).$$

This provides the proof for $p = r = 1$.

Let now $p \geq 2$. We notice that $t^2 + p/2 \alpha \geq t^2 + \alpha$. Then

$$g(t) \leq \frac{t(t^2 + \alpha)^{p/2-1} - t^{p-1}}{(t^2 + \alpha)^{p/2}}.$$

Setting $y = t^2/\alpha$, we have

$$\begin{aligned}
\int_0^q g(t) dt &\leq \int_0^{q^2/\alpha} \frac{(y+1)^{p/2-1} - y^{p/2-1}}{2(y+1)^{p/2}} dy \\
&\leq \frac{1}{2} \left(\frac{p}{2} - 1 \right) \int_0^{q^2/\alpha} \frac{1}{(y+1)^2} dy \\
&= \frac{1}{2} \left(\frac{p}{2} - 1 \right) \frac{q^2}{\alpha + q^2}.
\end{aligned}$$

We substitute this inequality into (2.43). We have

$$\begin{aligned}
f(\alpha) &\geq \alpha^{p/2} \left(\int_0^q \frac{t}{t^2 + p/2 \alpha} dt - \frac{1}{2} \left(\frac{p}{2} - 1 \right) \right) \\
&= \frac{\alpha^{p/2}}{2} \ln \left(\frac{q^2 + p/2 \alpha}{p/2 \alpha} \right) - \frac{\alpha^{p/2}}{2} \left(\frac{p}{2} - 1 \right) \frac{q^2}{\alpha + q^2}.
\end{aligned}$$

Since $q = \sqrt{1 - \sum_{i=p+1}^n b_i^2}$ and $p = r$, we obtain the lower bound

$$\begin{aligned}
[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq \frac{p c_p}{2 c_n} \int_{B_{n-p}} \alpha^{p/2} \ln \left(\frac{1 - \sum_{i=p+1}^n b_i^2 + p/2 \alpha}{p/2 \alpha} \right) db - \\
&\quad - \frac{p c_p}{2 c_n} \left(\frac{p}{2} - 1 \right) \int_{B_{n-p}} \alpha^{p/2} \frac{1 - \sum_{i=r}^n b_i^2}{\alpha + 1 - \sum_{i=r}^n b_i^2} db.
\end{aligned}$$

Let $\|b\|^2 = \sum_{i=p+2}^n b_i^2$, and $t = b_{p+1}/(1 - \|b\|^2)^{1/2}$. Then from the definition of α and using [4.642] of [GR94], we have

$$\begin{aligned}
[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \frac{p c_p}{c_n} \int_{B_{n-p-1}} \int_0^1 t^p (1 - \|b\|^2)^{(p+1)/2} \ln \left(\frac{1 - (1 - p/2 a^2) t^2}{p/2 a^2 t^2} \right) dt db - \\
&\quad - a^p \frac{p c_p}{c_n} \left(\frac{p}{2} - 1 \right) \int_{B_{n-p-1}} \int_0^1 t^p (1 - \|b\|^2)^{(p+1)/2} \frac{(1 - t^2)}{1 - (1 - a^2) t^2} dt db.
\end{aligned}$$

Using again [4.642] we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \gamma B\left(\frac{n-p-1}{2}, \frac{p+1}{2} + 1\right) \int_0^1 t^p \ln\left(\frac{1 - (1 - p/2 a^2)t^2}{p/2 a^2 t^2}\right) dt - \\ &\quad - a^p \gamma \frac{p-2}{2} B\left(\frac{n-p-1}{2}, \frac{p+1}{2} + 1\right) \int_0^1 \frac{t^p(1-t^2)}{1 - (1 - a^2)t^2} dt, \end{aligned} \quad (2.44)$$

where $\gamma = p(n-p-1)c_p c_{n-p-1}/(2c_n)$. Observe that

$$\begin{aligned} \int_0^1 t^p \ln\left(\frac{1 - (1 - p/2 a^2)t^2}{p/2 a^2 t^2}\right) dt &= \\ &= \frac{1}{p+1} \ln\left(\frac{2}{pa^2}\right) + \frac{2}{(p+1)^2} + \int_0^1 t^p \ln\left(1 - \left(1 - \frac{p}{2} a^2\right) t^2\right) dt. \end{aligned} \quad (2.45)$$

Notice that if $a^2 > 2/p$ then $\ln(1 - (1 - p/2 a^2)t^2) \geq \ln(1) = 0$. Hence, from (2.45) and using formula [3.197, 3] of [GR94] to solve the integral in (2.44), we have

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \frac{\gamma'}{p+1} \ln\left(\frac{2}{pa^2}\right) - \\ &\quad - a^p \gamma' \frac{2(p-2)}{(p+3)} F\left(1, \frac{p+1}{2}; \frac{p+5}{2}; 1 - a^2\right) + a^p \gamma' \frac{2}{p+1}, \end{aligned}$$

where $\gamma' = \gamma B((n-p-1)/2, (p+1)/2 + 1)$. Using (1.6), we can express c_i in terms of the gamma function, and we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \ln\left(\frac{1}{a^2}\right) - \\ &\quad - a^p \frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \left[\frac{2(p-2)}{p+3} F\left(1, \frac{p+1}{2}; \frac{p+5}{2}; 1 - a^2\right) - \ln\left(\frac{2}{p}\right) - \frac{2}{p+1} \right]. \end{aligned} \quad (2.46)$$

Otherwise, when $a^2 \leq 2/p$, we use the fact that

$$\ln(1 - ct^2) = - \sum_{i=1}^{\infty} \frac{(ct^2)^i}{i},$$

where c is a constant such that $-1 \leq ct^2 < 1$. Setting $c = (1 - p/2 a^2)$, and using previous relation, the integral in (2.45) becomes

$$\int_0^1 t^p \ln\left(1 - \left(1 - \frac{p}{2} a^2\right) t^2\right) dt = - \sum_{i=1}^{\infty} \frac{(1 - p/2 a^2)^i}{i(2i + p + 1)} \geq - \frac{1}{p+1} \ln\left(\frac{p+3}{2}\right).$$

In this case, from (2.45) we have

$$\int_0^1 t^p \ln\left(\frac{1 - (1 - p/2 a^2)t^2}{p/2 a^2 t^2}\right) dt \geq \frac{1}{p+1} \ln\left(\frac{2}{pa^2}\right) - \frac{1}{p+1} \ln\left(\frac{p+3}{2}\right) + \frac{2}{(p+1)^2},$$

and then

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^p \frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \ln\left(\frac{1}{a^2}\right) - \\ &\quad - a^p \frac{\Gamma((p+1)/2)}{\Gamma(p/2) \Gamma(1/2)} \left[\frac{2(p-2)}{p+3} F\left(1, \frac{p+1}{2}; \frac{p+5}{2}; 1 - a^2\right) + \ln\left(\frac{p+3}{2}\right) - \ln\left(\frac{2}{p}\right) - \frac{2}{p+1} \right], \end{aligned}$$

which concludes the proof for $p = r$.

The last case is $p > r$. Setting $y = t^2/\alpha$, the integral $f(\alpha)$ defined by (2.35) becomes

$$f(\alpha) = \frac{\alpha^{r/2}}{2} \int_0^{q^2/\alpha} \frac{y^{r/2-1}}{(y+1)^{p/2}} dy.$$

It can be rewritten as

$$f(\alpha) = \frac{\alpha^{r/2}}{2} \left[\int_0^\infty \frac{y^{r/2-1}}{(y+1)^{p/2}} dy - \int_{q^2/\alpha}^\infty \frac{y^{r/2-1}}{(y+1)^{p/2}} dy \right]. \quad (2.47)$$

The first integral of the right hand side of (2.47) can be solved using formula [3.194, 3] of [GR94] and is equal to $B(r/2, (p-r)/2)$. The second integral of (2.47) can be solved using formula [3.194, 2] of [GR94] and is equal to

$$\left(\frac{\alpha}{q^2}\right)^{(p-r)/2} \frac{2}{p-r} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -\frac{\alpha}{q^2}\right).$$

Hence, (2.47) becomes

$$f(\alpha) = \frac{\alpha^{r/2}}{2} B\left(\frac{r}{2}, \frac{p-r}{2}\right) - \frac{\alpha^{p/2}}{(p-r)q^{p-r}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -\frac{\alpha}{q^2}\right). \quad (2.48)$$

By substituting (2.48) into (2.34), and from the definition of α and q we have

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^r \frac{rc_r}{2c_n} B\left(\frac{r}{2}, \frac{p-r}{2}\right) \int_{B_{n-r}} b_{r+1}^r db - \\ &- a^p \frac{rc_r}{(p-r)c_n} \int_{B_{n-r}} \frac{b_{r+1}^p}{(1 - \sum_{i=r+1}^n b_i^2)^{(p-r)/2}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -\frac{a^2 b_{r+1}^2}{1 - \sum_{i=r+1}^n b_i^2}\right) db. \end{aligned}$$

Using again the technique of reducing integrals to one-dimensional integrals, we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p &\geq a^r \frac{\gamma}{2} B\left(\frac{r}{2}, \frac{p-r}{2}\right) B\left(\frac{r+1}{2}, \frac{n-r-1}{2} + 1\right) - \\ &- a^p \gamma' \int_0^1 \frac{t^p}{(1-t^2)^{(p-r)/2}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -a^2 \frac{t^2}{1-t^2}\right) dt, \end{aligned} \quad (2.49)$$

where $\gamma = rc_r c_{n-r-1}/c_n$ and $\gamma' = \Gamma((r+1)/2 + 1) / ((p-r)\Gamma(1/2)\Gamma(r/2))$. Consider the integral in (2.49). By setting $z = t^2/(1-t^2)$, we obtain

$$\begin{aligned} &\int_0^1 \frac{t^p}{(1-t^2)^{(p-r)/2}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -a^2 \frac{t^2}{1-t^2}\right) dt = \\ &= \frac{1}{2a^{p-r-2}} \int_0^\infty \frac{z^{(p-1)/2}}{(a^2+z)^{(r+3)/2}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -z\right) dz. \end{aligned} \quad (2.50)$$

We notice that

$$\frac{z^{(p-1)/2}}{(a^2+z)^{(r+3)/2}} \leq \frac{z^{(p-r)/2}}{(a^2+z)^2}.$$

Using this inequality and formula [7.51, 10] of [GR94], (2.50) can be bounded as follows

$$\begin{aligned} &\frac{1}{2a^{p-r-2}} \int_0^\infty \frac{z^{(p-1)/2}}{(a^2+z)^{(r+3)/2}} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -z\right) dz \leq \\ &\leq \frac{1}{2a^{p-r-2}} \int_0^\infty \frac{z^{(p-r)/2}}{(a^2+z)^2} F\left(\frac{p}{2}, \frac{p-r}{2}; \frac{p-r}{2} + 1; -z\right) dz \\ &= \frac{1}{2a^{p-r-2}} \frac{\Gamma((p-r)/2 + 1) \Gamma(r/2 + 1)}{\Gamma(p/2 + 1)} F\left(\frac{r}{2} + 1, 1; \frac{p}{2} + 1; 1 - a^2\right). \end{aligned} \quad (2.51)$$

Substituting (2.51) in (2.49), we get

$$\begin{aligned} [e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^{1/p} &\geq a^r \frac{\Gamma((p-r)/2) \Gamma((r+1)/2 + 1)}{\Gamma(p/2) \Gamma(1/2)} - \\ &- a^{r+2} \frac{r \Gamma((r+1)/2 + 1) \Gamma((p-r)/2)}{\Gamma(p/2 + 1) \Gamma(1/2)} F\left(\frac{r}{2} + 1, 1; \frac{p}{2} + 1; 1 - a^2\right). \end{aligned}$$

This concludes the proof. \square

2.4.3 Asymptotic Behavior

In Section 2.4.1 and 2.4.2 we provide upper and lower bounds for the randomized error of the power method at each step k . These bounds differ only by multiplicative constants and by lower order terms. We notice that only for upper bounds the constants depend on the size of the matrix, while for the lower bounds they depend only on p and r . Moreover, if A is a large matrix, the constants of the upper bound become huge. So, it is natural to ask if these constants are sharp. We answer this question by analyzing the asymptotic behavior of the randomized error $e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$.

Theorem 2.4.3 *Let A be a symmetric positive definite matrix, let $r, r < n$, and s denote the multiplicities of the two largest eigenvalues λ_1 and λ_{r+1} of A . Let $x_{r+1} = \lambda_{r+1}/\lambda_1$. Then for every p , $1 \leq p < \infty$, we have*

$$\lim_{k \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^k} = \left(\frac{\Gamma((r-p)/2) \Gamma((p+s)/2)}{\Gamma(r/2) \Gamma(s/2)} \right)^{1/p} \quad \text{for } p < r,$$

$$\lim_{k \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^k (2k)^{1/r} [\ln(1/x_{r+1})]^{1/r}} = \left(\frac{\Gamma((p+s)/2)}{\Gamma(p/2) \Gamma(s/2)} \right)^{1/p} \quad \text{for } p = r,$$

$$\lim_{k \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^{kr/p}} = \left(\frac{\Gamma((p-r)/2) \Gamma((r+s)/2)}{\Gamma(p/2) \Gamma(s/2)} \right)^{1/p} \quad \text{for } p > r.$$

Proof. From (2.28) we have

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) = \left(\frac{1}{c_n} \int_{B_n} \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^{p/2} db \right)^{1/p}.$$

We can rewrite the previous equation as follows

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) = \left(\frac{1}{c_n} \int_{B_n} \left(\left(\frac{x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2} \right)^{1/2} + r_k(b) \right)^p db \right)^{1/p},$$

where

$$r_k(b) = \left(\frac{\sum_{i=r+1}^n b_i^2 x_i^{2k}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 x_i^{2k}} \right)^{1/2} - \left(\frac{x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2} \right)^{1/2}. \quad (2.52)$$

Let

$$\tilde{e}_k^{\text{ran}}(A, p) = \left(\frac{1}{c_n} \int_{B_n} \left(\frac{x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2}{\sum_{i=1}^r b_i^2 + x_{r+1}^{2k} \sum_{i=r+1}^{r+s} b_i^2} \right)^{p/2} db \right)^{1/p}.$$

We want to show that

$$\lim_{k \rightarrow +\infty} e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) = \lim_{k \rightarrow +\infty} \tilde{e}_k^{\text{ran}}(A, p). \quad (2.53)$$

Notice that

$$\tilde{e}_k^{\text{ran}}(A, p) \leq e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) \leq \tilde{e}_k^{\text{ran}}(A, p) + \|r_k\|_p,$$

where

$$\|r_k\|_p = \left(\frac{1}{c_n} \int_{B_n} r_k(b)^p db \right)^{1/p}.$$

Since $r_k(b) \rightarrow 0$ pointwise almost everywhere, and $|r_k(b)| \leq 1$ for the \mathcal{L}_p -dominated convergence Theorem (see [Lan69], p. 312) we have $\lim_{k \rightarrow +\infty} \|r_k\|_p = 0$. This proves (2.53).

Equation (2.53) shows that the asymptotic behavior of $e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ can be studied by analyzing $\tilde{e}_k^{\text{ran}}(A, p)$. Let $a = x_{r+1}^k$. Integrating with respect to b_{r+s+1}, \dots, b_n , we have

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \frac{c_{n-r-s}}{c_n} \int_{B_{r+s}} \left(\frac{\sum_{i=r+1}^{r+s} b_i^2}{\sum_{i=1}^r b_i^2 + a^2 \sum_{i=r+1}^{r+s} b_i^2} \right)^{p/2} \left(1 - \sum_{i=1}^{r+s} b_i^2 \right)^{(n-r-s)/2} db$$

Let $\|b\|^2 = \sum_{i=1}^r b_i^2$ and let $t_i = b_i / (1 - \|b\|^2)^{1/2}$ for $i = r+1, \dots, r+s$, and $\|t\|^2 = \sum_{i=r+1}^{r+s} t_i^2$. If we rewrite the last integral as an integral over the balls B_r and B_s , we have

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \frac{c_{n-r-s}}{c_n} \int_{B_r} \int_{B_s} \frac{\|t\|^p (1 - \|b\|^2)^{(n+p-r)/2} (1 - \|t\|^2)^{(n-r-s)/2}}{[\|b\|^2 + a^2 \|t\|^2 (1 - \|b\|^2)]^{p/2}} dt db.$$

Using [4.642] of [GR94] for both integrals, we get

$$\begin{aligned} [\tilde{e}_k^{\text{ran}}(A, p)]^p &= \\ &= a^p \gamma \int_0^1 \int_0^1 \frac{t^{s-1} b^{r-1} t^p (1-b^2)^{(n+p-r)/2} (1-t^2)^{(n-r-s)/2}}{[b^2 + a^2 t^2 (1-b^2)]^{p/2}} dt db \\ &= a^p \gamma \int_0^1 t^{p+s-1} (1-t^2)^{(n-r-s)/2} \left[\int_0^1 \frac{b^{r-1} (1-b^2)^{(n+p-r)/2}}{[b^2 + a^2 t^2 (1-b^2)]^{p/2}} db \right] dt, \end{aligned} \quad (2.54)$$

where $\gamma = r s c_{n-r-s} c_r c_s / c_n$.

We have now three cases depending on the relation between p and r . Consider first the case $p < r$. Then the last integral of (2.54) is finite even for $a = 0$. Substituting $a = 0$, we get

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \gamma \int_0^1 t^{p+s-1} (1-t^2)^{(n-r-s)/2} dt \int_0^1 b^{r-p-1} (1-b^2)^{(n+p-r)/2} db.$$

From the definition of the beta function (1.7) we have

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \frac{\gamma}{4} B\left(\frac{p+s}{2}, \frac{n-r-s}{2} + 1\right) B\left(\frac{r-p}{2}, \frac{n+p-r}{2} + 1\right).$$

Using (1.6), we can express c_i in terms of the gamma function. We obtain

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \frac{\Gamma((r-p)/2) \Gamma((p+s)/2)}{\Gamma(r/2) \Gamma(s/2)}.$$

This proves that for $p < r$, by using (2.53) we have

$$\lim_{k \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^k} = \left(\frac{\Gamma((r-p)/2) \Gamma((p+s)/2)}{\Gamma(r/2) \Gamma(s/2)} \right)^{1/p}.$$

Consider now the case $p = r$. From (2.54) we have

$$[\tilde{e}_k^{\text{ran}}(A, p)]^p = a^p \gamma \int_0^1 t^{p+s-1} (1-t^2)^{(n-p-s)/2} \left[\int_0^1 \frac{b^{p-1} (1-b^2)^{n/2}}{[b^2 + a^2 t^2 (1-b^2)]^{p/2}} db \right] dt, \quad (2.55)$$

We expand $b^{p-1} (1-b^2)^{n/2}$ as $b^{p-1} - (n/2)b^{p+1} + O(b^{p+3})$. Since $[b^2(1-a^2t^2) + a^2t^2]^{p/2}$ behaves as $b^p + o(a^2t^2)$, it is sufficient to consider the first two terms of the expansion. As a approaches zero, we have

$$\begin{aligned} & \int_0^1 \frac{b^{p-1} (1-b^2)^{n/2}}{[b^2(1-a^2t^2) + a^2t^2]^{p/2}} db = \\ &= \int_0^1 \frac{b^{p-1}}{[b^2(1-a^2t^2) + a^2t^2]^{p/2}} db + O\left(\int_0^1 \frac{b^{p+1}}{[b^2(1-a^2t^2) + a^2t^2]^{p/2}} db\right) \\ &= \int_0^1 \frac{b^{p-1}}{(b^2 + a^2t^2)^{p/2}} db + O\left(\int_0^1 b db\right) \end{aligned}$$

Observe that $(b^2 + a^2t^2)^{p/2} = b^p + (p/2)b^{2(p/2-1)}a^2t^2(1+o(1))$ as $a \rightarrow 0$. Then from the last equation we have

$$\begin{aligned} & \int_0^1 \frac{b^{p-1}}{(b^2 + a^2t^2)^{p/2}} db + O\left(\int_0^1 b db\right) = \\ &= \int_0^1 \frac{b^{p-1}}{b^{p-2}(b^2 + p/2 a^2t^2)} db + O(1) \\ &= \int_0^1 \frac{b}{b^2 + p/2 a^2t^2} db + O(1) \\ &= \frac{1}{2} \ln\left(b^2 + \frac{p}{2} a^2t^2\right) \Big|_0^1 + O(1) \\ &= \ln\left(\sqrt{\frac{2}{pa^2t^2}}\right) (1+o(1)). \end{aligned}$$

Substituting this equality into (2.55) we get

$$\begin{aligned} [\tilde{e}_k^{\text{ran}}(A, p)]^p &= a^p \gamma \int_0^1 t^{p+s-1} (1-t^2)^{(n-p-s)/2} \ln\left(\sqrt{\frac{2}{pa^2t^2}}\right) dt \\ &= a^p \frac{\gamma}{4} \ln\left(\frac{2}{pa^2}\right) B\left(\frac{p+s}{2}, \frac{n-p-s}{2} + 1\right) + O(a^p). \end{aligned}$$

If we replace the expression for γ in the last equation, from (2.53) we obtain

$$\lim_{k \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^k (2k)^{1/r} [\ln(1/x_{r+1})]^{1/r}} = \left(\frac{\Gamma((p+s)/2)}{\Gamma(p/2)\Gamma(s/2)}\right)^{1/p}.$$

The last case is $p > r$. We want to compute the limit

$$\lim_{x \rightarrow +\infty} \frac{e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)}{x_{r+1}^{kr/p}} = \left[\lim_{k \rightarrow +\infty} \frac{[\tilde{e}_k^{\text{ran}}(A, p)]^p}{x_{r+1}^{kr}} \right]^{1/p}$$

From (2.54) we get

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \frac{[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p}{x_{r+1}^{kr}} = \lim_{a \rightarrow 0} \frac{[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p}{a^r} = \\ &= \lim_{a \rightarrow 0} a^{p-r} \gamma \int_0^1 t^{p+s-1} (1-t^2)^{(n-r-s)/2} \left[\int_0^1 \frac{b^{r-1} (1-b^2)^{(n+p-r)/2}}{[b^2 + a^2 t^2 (1-b^2)]^{p/2}} db \right] dt. \end{aligned} \quad (2.56)$$

Observe that for $a \rightarrow 0$ we have

$$\begin{aligned}
& \int_0^1 a^{p-r} \frac{b^{r-1}(1-b^2)^{(n+p-r)/2}}{[b^2 + a^2 t^2(1-b^2)]^{p/2}} db = \\
& = \int_0^1 a^{p-r} \frac{b^{r-1}(1-b^2)^{(n+p-r)/2}}{[b^2 + a^2 t^2]^{p/2}} db \\
& = \int_0^1 a^{p-r} \frac{b^{r-1}(1-b^2)^{(n+p-r)/2}}{a^p t^p (b^2/(a^2 t^2) + 1)^{p/2}} db.
\end{aligned} \tag{2.57}$$

We change variables by setting $y = b/(at)$. Then the integral (2.57) becomes

$$\frac{1}{t^{p-r}} \int_0^{1/(at)} \frac{y^{r-1}(1-a^2 t^2 y^2)^{(n+p-r)/2}}{(y^2 + 1)^{p/2}} dy.$$

If we set $z = y^2$, this integral can be transformed into

$$\frac{1}{2t^{p-r}} \int_0^{1/(a^2 t^2)} \frac{z^{r/2-1}(1-a^2 t^2 z)^{(n+p-r)/2}}{(z+1)^{p/2}} dz.$$

We substitute this integral into (2.56). We get

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \frac{[\tilde{e}_k^{\text{ran}}(A, p)]^p}{x_{r+1}^{kr}} = \lim_{a \rightarrow 0} \frac{[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p}{a^r} = \\
& = \frac{\gamma}{2} \int_0^1 t^{r+s-1}(1-t^2)^{\frac{n-r-s}{2}} \left[\lim_{a \rightarrow 0} \int_0^{\frac{1}{a^2 t^2}} \frac{z^{\frac{r}{2}-1}(1-a^2 t^2 z)^{\frac{n+p-r}{2}}}{(z+1)^{\frac{p}{2}}} dz \right] dt.
\end{aligned} \tag{2.58}$$

To find the limit of the last integral, we use the following bounds (for $a < 1$)

$$\int_0^{\frac{1}{at}} \frac{z^{\frac{r}{2}-1}(1-at)^{\frac{n+p-r}{2}}}{(z+1)^{\frac{p}{2}}} dz \leq \int_0^{\frac{1}{a^2 t^2}} \frac{z^{\frac{r}{2}-1}(1-a^2 t^2 z)^{\frac{n+p-r}{2}}}{(z+1)^{\frac{p}{2}}} dz \leq \int_0^{\frac{1}{a^2 t^2}} \frac{z^{\frac{r}{2}-1}}{(z+1)^{\frac{p}{2}}} dz.$$

Since

$$\lim_{a \rightarrow 0} \int_0^{1/(at)} \frac{z^{r/2-1}(1-at)^{(n+p-r)/2}}{(z+1)^{p/2}} dz = \lim_{a \rightarrow 0} \int_0^{1/(a^2 t^2)} \frac{z^{r/2-1}}{(z+1)^{p/2}} dz,$$

passing to the limit and then using [3.194, 3] of [GR94], we get

$$= \lim_{a \rightarrow 0} \int_0^{1/(a^2 t^2)} \frac{z^{r/2-1}}{(z+1)^{p/2}} dz = \int_0^{+\infty} \frac{z^{r/2-1}}{(z+1)^{p/2}} dz = B\left(\frac{r}{2}, \frac{p-r}{2}\right).$$

Hence, we also have

$$\lim_{a \rightarrow 0} \int_0^{1/(a^2 t^2)} \frac{z^{r/2-1}(1-a^2 t^2 z)^{(n+p-r)/2}}{(z+1)^{p/2}} dz = B\left(\frac{r}{2}, \frac{p-r}{2}\right).$$

From (2.58), we get

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \frac{[e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)]^p}{x_{r+1}^{kr}} = \\
& = \frac{\gamma}{2} B\left(\frac{r}{2}, \frac{p-r}{2}\right) \int_0^1 t^{r+s-1}(1-t^2)^{(n-r-s)/2} dt \\
& = \frac{\Gamma((p-r)/2) \Gamma((r+s)/2)}{\Gamma(p/2) \Gamma(s/2)}.
\end{aligned}$$

This concludes the proof. \square

Theorem 2.4.3 shows that upper and lower bounds provided in Section 2.4.1 and 2.4.2 are asymptotically optimal. In fact, the analysis of the asymptotic case indicates that the upper and lower bounds cannot be improved since the constants coincide with those of the upper bound when we set the multiplicity of the second largest eigenvalue to $n - r$, and with those of the lower bound for $s = 1$. The constants increase with s and $1/r$. This corresponds to the intuitive idea that the convergence is fast if the eigenspace \mathbf{Z}_1 is large, and is slow if the eigenspace corresponding to the second largest eigenvalue is large. Note that if p approaches infinity, the rate of convergence approaches 1 and even the constant converges to 1. This agrees with (2.30) for $p = \infty$.

2.5 Discussion

Theorems 2.2.3 and 2.2.4 as well as Theorems 2.4.1 and 2.4.2 show how the randomized errors depend on the ratio λ_{r+1}/λ_1 . In addition, these theorems describe the actual behavior of the rate of convergence for every k , p and r . We notice that only when $p < r$ (respectively $2p < r$ for eigenvalue estimate), we have the same rate of convergence as in the asymptotic deterministic case with $\sum_{i=1}^r b_i^2 \neq 0$. For the other two cases, $p = r$ and $p > r$ ($2p = r$ and $2p > r$ respectively for eigenvalue), the rate convergence is slower. This is due to the fact that these theorems deal with the randomized case. So, in order to compute the randomized error we have to integrate over all possible starting vectors, even those for which the power method does not converge or converges very slowly.

To give an intuitive idea about the difference in the rate of convergence between the asymptotic deterministic case and the randomized case, let us analyze the error for $p = 1$ for the eigenvector approximation. In this case we have only two possibilities: $r > p$ or $r = p = 1$. Assuming $\sum_{i=1}^r b_i^2 \neq 0$, we have

$$\inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\text{Pow}} - \mathbf{v}\| = \sin(\alpha_k(\mathbf{b})) = \left(\frac{\lambda_{r+1}}{\lambda_1}\right)^k \sqrt{\frac{b_{r+1}^2 + \dots + b_{r+s}^2}{b_1^2 + \dots + b_r^2}} + o\left(\left(\frac{\lambda_{r+1}}{\lambda_1}\right)\right),$$

where s is the multiplicity of the second largest eigenvalue.

If $r = 1$, the expected value of $\sin(\alpha_k(\mathbf{b}))$ with respect to \mathbf{b} cannot be proportional to $(\lambda_2/\lambda_1)^k$ since

$$\int_{\|\mathbf{b}\|=1} \sqrt{\frac{b_2^2 + \dots + b_{s+1}^2}{b_1^2}} \mu(db) = +\infty.$$

A more careful analysis shows that we have to lose a factor proportional to $\ln(\lambda_1/\lambda_2)^{2k}$ in order to achieve the convergence of the integral. For $r \geq 2$,

$$\int_{\|\mathbf{b}\|=1} \sqrt{\frac{b_{r+1}^2 + \dots + b_{r+s}^2}{b_1^2 + \dots + b_r^2}} \mu(db) < +\infty,$$

so we have a rate of convergence proportional to $(\lambda_{r+1}/\lambda_1)^k$ as in the deterministic case. The explanation of the general case $p \geq 1$ is similar.

Analyzing together upper and lower bounds we have a complete behavior of the power method for computing a largest eigenvector. In fact, for every p and r , upper and lower bounds exhibit the same dependence on λ_{r+1}/λ_1 and on k .

We now comment on the bounds proposed by Kostlan in [Kos88].

Kostlan estimates the number of steps required by the power method to give a dominant ε -eigenvector, averaged over all the possible starting vector. However, he considers another error criterion, so that it is not easy to compare these bounds with our bounds. In particular, we use the Euclidean distance where in [Kos88] the Rimanian distance is considered. Moreover, we study the error in the \mathcal{L}_p case, while Kostlan simply integrate the error over the all possible starting vectors.

2.6 Numerical Experiments

We tested the power method for several matrices with many pseudo-random starting vectors \mathbf{b} . The matrix A can be chosen as follows. As before, let $\xi_k^{\text{Pow}}(A, \mathbf{b})$ and $\mathbf{u}_k^{\text{Pow}}(A, \mathbf{b})$ be the approximation of λ_1 after k steps of the power method and the vector computed by the power method applied to the matrix A with starting vector \mathbf{b} respectively. Observe that for any orthogonal matrix Q , we have $\mathbf{u}_k(Q^T A Q, Q^T \mathbf{b}) = \mathbf{u}_k(A, \mathbf{b})$. Moreover, the uniform distribution on the unit sphere of the vectors \mathbf{b} implies the same distribution of vectors $Q^T \mathbf{b}$. So, without loss of generality, we can restrict ourselves only to consider diagonal matrices, see also [KW92a] and [KW94]. Vectors uniformly distributed over the unit sphere can be generated as described in [Knu81], [KW92a] and [KW94].

The tests were performed on a Sun SPARCsystem 10 using double precision. To compute the values of the hypergeometric and the gamma functions we used the program *Mathematica*.

We tested many different matrices of size 100 with the distributions of the eigenvalues chosen as in [KW94]. We tested the following distributions:

- Chebyshev distribution: $\lambda_1 = \dots = \lambda_r = 1 + \cos(\pi/200)$, $\lambda_{r+i} = 1 + \cos(((2i-1)\pi)/200)$;
- quadratic distribution 1: $\lambda_1 = \dots = \lambda_r = 2(1 - 1/101)^2$, $\lambda_{r+i} = 2(1 - i/101)^2$;
- quadratic distribution 2: $\lambda_1 = \dots = \lambda_r = 2(1 - (1/101)^2)$, $\lambda_{r+i} = 2(1 - (i/101)^2)$;
- uniform distribution: $\lambda_1 = \dots = \lambda_r = 2(1 - 1/101)$, $\lambda_{r+i} = 2(1 - i/101)$;
- logarithmic distribution: $\lambda_1 = \dots = \lambda_r = 2 \log(101)/\log(102)$, $\lambda_{r+i} = 2 \log(102 - i)/\log(102)$;
- exponential distribution 1: $\lambda_1 = \dots = \lambda_r = 2e^{-1}$; $\lambda_{r+i} = 2e^{-\sqrt[3]{i}}$;
- exponential distribution 2: $\lambda_1 = \dots = \lambda_r = 1 + e^{-1}$, $\lambda_{r+i} = 1 + e^{-i}$.

From the theoretical bounds, see Theorems 2.2.3, 2.2.4, 2.4.1 and 2.4.2, it turns out that the behavior of the power method depends on the relation between r and p . We tested the power method for different values of p and r for a fixed ratio between the two largest eigenvalues.

The main goal of these tests was to verify the upper and lower bounds and to see how much they differ from the experimental values.

In order to approximate the randomized errors $e^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$ and $e^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ we have used 1,000 pseudo-random vectors \mathbf{b} generated in accordance with the procedure described in Section 1.1.

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	p
10	$2.456e-02$	$1.466e-04$	$1.216e+01$	$9.737e-01$	$4.782e-01$	$7.998e+00$	1
100	$2.857e-03$	$1.423e-04$	$1.184e+01$	$9.111e-01$	$4.850e-01$	$7.992e+00$	1
1000	$3.457e-04$	$1.096e-04$	$9.086e+00$	$7.114e-01$	$5.185e-01$	$7.685e+00$	1
10	$2.512e-02$	$1.039e-04$	$2.441e+00$	$9.735e-01$	$6.457e-01$	$3.522e+00$	2
100	$3.156e-03$	$1.035e-04$	$2.409e+00$	$9.239e-01$	$6.394e-01$	$3.474e+00$	2
1000	$4.127e-04$	$9.893e-05$	$2.110e+00$	$7.383e-01$	$5.799e-01$	$3.035e+00$	2
10	$2.951e-02$	$6.440e-05$	$1.065e+00$	$9.779e-01$	$2.712e-01$	$1.129e+00$	10
100	$4.456e-03$	$6.503e-05$	$1.063e+00$	$9.412e-01$	$2.729e-01$	$1.127e+00$	10
1000	$8.194e-04$	$7.120e-05$	$1.035e+00$	$8.675e-01$	$2.902e-01$	$1.097e+00$	10

Table 2.1: Quadratic distribution 2.

So, the randomized errors are replaced by the mean values among the 1,000 pseudo-random vectors, i.e.

$$\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p) = \left(\frac{1}{1,000} \sum_{i=1}^{1,000} \frac{\lambda_1 - \xi_k^{\text{Pow}}}{\lambda_1} \right)^{1/p}, \quad (2.59)$$

$$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p) = \left(\frac{1}{1,000} \sum_{i=1}^{1,000} \sin^p(\alpha_k(\mathbf{b}_i)) \right)^{1/p}. \quad (2.60)$$

Let $\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$ denote the lower and upper bound computed in accordance with theorems 2.2.4 and 2.2.3 respectively, and let $\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ denote the lower and the upper bounds computed using formulas given by Theorems 2.4.2 and 2.4.1. Finally, k and p are the number of iterations and the parameter of the norm, respectively. Each table contains both randomized errors for eigenvalue and eigenvector approximation. In the first column is reported the number k of iterations, the second column contains the value $\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$ defined in equation (2.59), third and fourth column contain the theoretical upper and lower bounds $\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$. In the right hand side of the table we report the same quantities referred to eigenvector approximation. In the last column we report the value of the norm parameter p .

In order to underline the dependence of the rate of convergence on the ratio between the two largest eigenvalues we report the results obtained for the quadratic distribution 2, see Table 2.1, and the exponential distribution 1, see Table 2.2. In fact, these distributions are those (among the different distributions considered) for which we have the largest (the smallest) ratio between λ_2 and λ_1 and then the slowest (the fastest) convergence, respectively.

From Table 2.1 we see that for three different values of p , even after 1,000 iterations the randomized error for the approximation of the dominant eigenvector is still very close to 1, for the eigenvalue estimate the error is lower. An important observation concerns the lower and upper bounds. We notice that the lower bounds are good approximations of the expected values $\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ while the upper bounds are clearly overestimates. This is due to the following reasons:

1. The constants in the upper bounds, see Theorem 2.2.3 and 2.4.1, grow with the size of the matrix.

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	p
10	$1.812e-02$	$8.507e-03$	$9.004e-01$	$1.770e-01$	$1.698e-01$	$2.124e+00$	1
30	$1.390e-04$	$4.700e-05$	$4.975e-03$	$2.432e-03$	$2.300e-03$	$2.864e-02$	1
10	$5.161e-02$	$3.934e-02$	$6.621e-01$	$2.509e-01$	$2.368e-01$	$9.616e-01$	2
30	$5.270e-03$	$3.276e-03$	$4.921e-02$	$2.468e-02$	$2.006e-02$	$7.148e-02$	2
10	$1.532e-01$	$8.194e-02$	$8.164e-01$	$6.801e-01$	$3.888e-01$	$8.715e-01$	10
30	$1.150e-01$	$7.455e-02$	$4.854e-01$	$3.562e-01$	$3.421e-01$	$5.182e-01$	10

Table 2.2: Exponential distribution 1 with $r = 1$.

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$	p
10	$5.171e-02$	$1.107e-02$	$1.823e+00$	$4.100e-01$	$1.276e-01$	$1.920e+00$	1
30	$3.117e-05$	$1.525e-05$	$2.427e-03$	$3.765e-03$	$3.693e-03$	$4.622e-02$	1
10	$7.429e-02$	$9.315e-03$	$2.921e-01$	$4.593e-01$	$1.570e-01$	$3.650e+00$	2
30	$8.112e-04$	$4.103e-04$	$7.031e-03$	$7.979e-03$	$7.511e-03$	$1.245e-01$	2
10	$1.663e-01$	$3.632e-02$	$2.195e-01$	$6.904e-01$	$2.472e-01$	$8.850e-01$	10
30	$3.251e-02$	$3.004e-02$	$1.042e-01$	$2.551e-01$	$1.950e-01$	$4.200e-01$	10

Table 2.3: Exponential distribution 2 with $r = 2$.

- Since the ratio $x_2 = \lambda_2/\lambda_1$ is very close to 1, x_2^k goes very slowly to 0 with k . In this case, the upper bound is more sensitive of the big multiplicative constants.

Table 2.2 is more interesting since it allows us to see the dependence of the speed of convergence on p and r . The speed of convergence is now good. In fact, after only 30 iterations we get an error of the order of 10^{-4} for eigenvalue and 10^{-3} for eigenvector when $p = r = 1$. In this case, we have also that the upper and lower bounds are relatively close to each other, and that the randomized error for $k = 30$ is very close to the theoretical bounds.

In general, it is possible to observe that the values of $\varepsilon^{\text{ran}}(\xi_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ computed with these tests are very close to the theoretical lower bounds while they are more distant from the upper bounds even for small λ_{r+1}/λ_1 . This is due to the importance of the multiplicity s of λ_{r+1} , as it turns out from the asymptotic constants of Theorem 2.4.3. Experimental results prove that the power method behaves differently for matrices with the same two largest eigenvalues but with different multiplicities. In particular, increasing s we get bounds closer to the upper bounds.

To understand the role of p and r , we have performed tests with matrices for which the multiplicity of the largest eigenvalue is $r \geq 2$. In Table 2.3 we report the results for the exponential distribution 2 with $r = 2$.

An important observation concerns the comparison between the three cases, $2p < r$, $2p = r$ and $2p > r$ (respectively $p < r$, $p = r$ and $p > r$ for eigenvector). From Table 2.3 it is easy to see that for the same value of k , the rates of convergence are different. For example, for $k = 30$ we have that $\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ is of the order of 10^{-3} for $p \leq r$, and of order 10^{-1} for $p > r$.

We performed also tests with matrices with only two distinct eigenvalues. These tests indicate

the asymptotic dependence of the randomized error on the multiplicity s of the second eigenvalue. In particular, they show that the randomized errors are closer to the upper bounds $\varepsilon^{\text{ub}}(\xi_k^{\text{Pow}}, A, p)$ and $\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Pow}}, A, p)$ when s is big. This is an important consequence of Theorem 2.4.3.

Chapter 3

The Lanczos and Polynomial Methods

3.1 Worst Case Results

This chapter is devoted to the study of the randomized error of the Lanczos method and of other polynomial methods (see Chapter 1 for definitions).

The analysis of the Lanczos method for eigenvalue and eigenvector estimate is much harder than that of the power method, even in the deterministic case.

Saad [Saa80] proved upper bounds for the Lanczos method in the symmetric case. For completeness we report here two theorems. Theorem 3.1.1 refers to the eigenvalue approximation, Theorem 3.1.2 to eigenvector estimate. The proofs can be found in [Saa80] or in [Saa92].

Theorem 3.1.1 [Saa92] p. 201 *The relative error of estimating the largest eigenvalue λ_1 by the k -th approximation ξ_k^{Lan} returned by the Lanczos method satisfies the following inequality*

$$\frac{\lambda_1 - \xi_k^{\text{Lan}}}{\lambda_1} \leq \left(1 - \frac{\lambda_n}{\lambda_1}\right) \left(\frac{\tan \theta(\mathbf{b}, \mathbf{Z}_1)}{T_k(1 + 2\rho)}\right)^2,$$

where T_k is the k -th Chebyshev polynomial of first kind, $\theta(\mathbf{b}, \mathbf{Z}_1)$ is the angle between the starting vector \mathbf{b} and the eigenspace corresponding to λ_1 , and

$$\rho = \frac{\lambda_1 - \lambda_{r+1}}{\lambda_{r+1} - \lambda_n}.$$

□

Theorem 3.1.2 [Saa92] p. 203 *Let $\mathbf{u}_k^{\text{Lan}}$ be the vector returned by the Lanczos method after k steps as approximation of an eigenvector in \mathbf{Z}_1 . Let $\xi_i^{(k)}$, $i = 1, 2, \dots, k$, be the eigenvalues of the matrix $Q_k^T A Q_k$, where Q_k is an orthogonal basis of \mathcal{K}_k . Let $\theta(\mathbf{u}_k^{\text{Lan}}, \mathbf{Z}_1)$ be the acute angle between the vector $\mathbf{u}_k^{\text{Lan}}$ and the eigenspace \mathbf{Z}_1 . Then we have the following bound*

$$\inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\text{Lan}} - \mathbf{v}\| \leq \frac{\sqrt{1 + \gamma_{k+1}^2 / \delta}}{T_k(1 + 2\rho)} \tan \theta(\mathbf{b}, \mathbf{Z}_1),$$

where $\delta = \min_{i \neq 1} \|\lambda_1 - \xi_i^{(k)}\|$, and $\gamma_{k+1} = \|(I - R_k)AR_k\|$ where R_k is the orthogonal projector onto \mathcal{K}_k .

□

These two theorems show that the error in the estimate of λ_1 and of an eigenvector in the space \mathbf{Z}_1 , depends on the eigenvalues of the matrix and on the starting vector \mathbf{b} . In particular, when the starting vector is orthogonal to \mathbf{Z}_1 , the bounds are infinity. Moreover, the dependence on k is through the Chebyshev polynomial computed at the value $1 + 2\rho$. We see that, when $\rho = (\lambda_1 - \lambda_{r+1})/(\lambda_{r+1} - \lambda_n)$ goes to zero, the argument of the polynomial T_k goes to 1 and that the bounds do not decrease with k . This shows that the speed of convergence of the Lanczos method depends on the particular distribution of the eigenvalues.

The rest of this chapter is organized as follows. Section 3.2 reports results about eigenvector estimate. We first comment about the impossibility to bound the randomized error with a quantity that goes to zero when $k < n$ but that is independent of the particular matrix. This result holds for every polynomial method. We then give distribution dependent bounds. In Section 3.4 we report distribution free bounds as well as distribution dependent bounds for eigenvalue estimate. Finally, Section 3.5 contains numerical experiments.

The result of this chapter are mainly contained in [dCM96].

3.2 Best Eigenvector Approximation

In this section we analyze the behavior of the best polynomial method \mathcal{B} introduced in Section 1.1, equation (1.3). The main result of this section is the computation of the supremum of the \mathcal{L}_p randomized error over all positive definite matrices. In Chapter 2 the same problem has been analyzed for the power method. It is shown that there are matrices for which the distribution of the eigenvalues is so bad that the power method does not converge, even for a random starting vector. In [LW96] the analysis is generalized to all polynomial methods showing that there exists an $n \times n$ matrix for which all polynomial methods fail in the approximation of an eigenvector in \mathbf{Z}_1 , unless n steps are performed. This “worst case” matrix has a unique largest eigenvalue but the others are pathologically close to λ_1 . Hence, polynomial methods are not able to distinguish between the eigenspace \mathbf{Z}_1 and the eigenspace corresponding to the second largest eigenvalue. In [LW96] the randomized error is considered only in the \mathcal{L}_2 case. The goal of this section is to generalize this result to the \mathcal{L}_p case, for $1 \leq p < +\infty$. In addition, we prove an upper bound on the randomized error of the method \mathcal{B} which shows how the distribution of the eigenvalues affects the speed of convergence.

Theorem 3.2.1 *Let \mathcal{A}_n denote the class of all $n \times n$ positive definite matrices.*

a) *For $k < n$ we have*

$$\sup_{A \in \mathcal{A}_n} e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) = \left(\frac{\Gamma((n-k+1)/2) \Gamma((n+p-k)/2)}{\Gamma((n-k)/2) \Gamma((n+p-k+1)/2)} \right)^{1/p}.$$

b) *For $k \geq n$ the method \mathcal{B} has zero randomized error.*

Let us comment on this theorem. Observe that the value

$$\left(\frac{\Gamma((n-k+1)/2) \Gamma((n+p-k)/2)}{\Gamma((n-k)/2) \Gamma((n+p-k+1)/2)} \right)^{1/p} \quad (3.1)$$

is decreasing with k since $k > h$ implies $\mathcal{K}_h \subset \mathcal{K}_k$. For $k = n - 1$ the value in (3.1) becomes

$$\left(\frac{\Gamma((p+1)/2)}{\Gamma(1/2) \Gamma(p/2+1)} \right)^{1/p} = \left(\frac{2\Gamma((p+1)/2)}{\pi^{1/2} p \Gamma(p/2)} \right)^{1/p}. \quad (3.2)$$

Hence, even for $k = n - 1$ there are matrices for which the best polynomial method \mathcal{B} fails.

Proof of Theorem 3.2.1. Part b) follows by observing that $e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = 0$ (it will be proven in Theorem 3.3.1), and $e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) \leq e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$.

To prove ¹ part a), assume $k < n$, and let r denote the multiplicity of the largest eigenvalue λ_1 . Let $\mathbf{b} = \sum_{i=1}^n b_i \mathbf{z}_i$. An arbitrary vector $\mathbf{u}_k \in \mathcal{K}_k$ with $\|\mathbf{u}_k\| = 1$ can be written as

$$\mathbf{u}_k = \frac{\sum_{i=1}^n b_i P(\lambda_i) \mathbf{z}_i}{\sqrt{\sum_{i=1}^n b_i^2 P^2(\lambda_i)}},$$

where P is a polynomial of degree at most $k - 1$. We have

$$\inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k - \mathbf{v}\| = \left(\frac{\sum_{i=r+1}^n b_i^2 P^2(\lambda_i)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)} \right)^{1/2}. \quad (3.3)$$

Let \mathcal{P}_k denote the class of all polynomials of degree at most $k - 1$. By (1.3) and (3.3) we have that $\mathbf{u}_k^{\mathcal{B}}$ satisfies

$$\inf_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\mathcal{B}} - \mathbf{v}\| = \min_{P \in \mathcal{P}_k} \left(\frac{\sum_{i=r+1}^n b_i^2 P^2(\lambda_i)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)} \right)^{1/2}. \quad (3.4)$$

By (1.4) we have,

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) = \left(\int_{\|b\|=1} \min_{P \in \mathcal{P}_k} \left(\frac{\sum_{i=r+1}^n b_i^2 P^2(\lambda_i)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)} \right)^{p/2} \mu(db) \right)^{1/p}.$$

Using a continuity argument we may restrict ourselves to polynomials P that are not equal to zero in λ_1 . Let $Q(t) = P(t\lambda_1)/P(\lambda_1)$. The polynomial Q belongs to \mathcal{P}_k and we have that $Q(1) = 1$. Let $\mathcal{P}_k(1)$ denote the class of polynomial of degree at most $k - 1$ that takes 1 at 1. For $x_i = \lambda_i/\lambda_1$, from the previous equality we have

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) = \left(\int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^{p/2} \mu(db) \right)^{1/p}.$$

Let Q^* be the polynomial in $\mathcal{P}_k(1)$ that minimizes the integrand value. We want to prove that

$$\sup_{A \in \mathcal{A}_n} \left(\int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^*(x_i)^2} \right)^{p/2} \mu(db) \right)^{1/p} = \left(\int_{\|b\|=1} \left(\frac{\sum_{i=2}^{n-k+1} b_i^2}{b_1^2 + \sum_{i=2}^{n-k+1} b_i^2} \right)^{p/2} \mu(db) \right)^{1/p}. \quad (3.5)$$

¹The first part of this proof is a generalization of the \mathcal{L}_2 proof due to Leyk and Woźniakowski [LW96].

In order to prove (3.5) we first note that for the worst possible Equality (3.5) can be proven by simply applying the same reasoning of [LW96] to the \mathcal{L}_p case.

The problem is then reduced to the solution of the following integral

$$S = \int_{\|b\|=1} \left(\frac{\sum_{i=2}^{n-k+1} b_i^2}{b_1^2 + \sum_{i=2}^{n-k+1} b_i^2} \right)^{p/2} \mu(db). \quad (3.6)$$

Since the integrand function satisfies the assumptions of equality (1.9), we have

$$S = \frac{1}{c_n} \int_{B_n} \left(\frac{\sum_{i=2}^{n-k+1} b_i^2}{b_1^2 + \sum_{i=2}^{n-k+1} b_i^2} \right)^{p/2} db. \quad (3.7)$$

Integrating with respect to the last $k-1$ variables, we get

$$S = \frac{c_{k-1}}{c_n} \int_{B_{n-k+1}} \left(\frac{\sum_{i=2}^{n-k+1} b_i^2}{b_1^2 + \sum_{i=2}^{n-k+1} b_i^2} \right)^{p/2} \left(1 - \sum_{i=1}^{n-k+1} b_i^2 \right)^{(k-1)/2} db \quad (3.8)$$

Rewriting last integral as an integral over the balls $B_1 = \{b_1, b_1^2 \leq 1\}$ and the $(n-k)$ -dimensional ball B' of radius $\sqrt{1-b_1^2}$, we get

$$S = \frac{c_{k-1}}{c_n} \int_{B_1} \int_{B'} \frac{\|b\|^p}{(b_1^2 + \|b\|^2)^{p/2}} (1 - b_1^2 - \|b\|^2)^{(k-1)/2} db, \quad (3.9)$$

where $\|b\|^2 = \sum_{i=2}^{n-k+1} b_i^2$.

Let $t_i = b_i/\sqrt{1-b_1^2}$, for $i = 2, \dots, n-k+1$, and let $\|t\|^2 = \sum_{i=2}^{n-k+1} t_i^2 = \|b\|^2/(1-b_1^2)$. We rewrite equation (3.9) as follows

$$S = \frac{c_{k-1}}{c_n} \int_{B_1} \int_{B_{n-k}} \frac{\|t\|^p (1-b_1^2)^{p/2} (1-b_1^2)^{(k-1)/2} (1-t^2)^{(k-1)/2} (1-b_1^2)^{(n-k)/2}}{(b_1^2 + \|t\|^2 (1-b_1^2))^{p/2}} dt db_1. \quad (3.10)$$

Applying two times formula [4.642] of [GR94] we get

$$\begin{aligned} S &= \gamma \int_0^1 \int_0^1 \frac{t^{n-k+p-1} (1-t^2)^{(k-1)/2} (1-b^2)^{(n+p-1)/2}}{(b^2 + t^2(1-b^2))^{p/2}} dt db \\ &= \gamma \int_0^1 t^{n-k+p-1} (1-t^2)^{(k-1)/2} \int_0^1 \frac{(1-b^2)^{(n+p-1)/2}}{(b^2 + t^2(1-b^2))^{p/2}} db dt \end{aligned} \quad (3.11)$$

where $\gamma = 2c_{k-1}(n-k)c_{n-k}/c_n$.

Setting $z = (1-b^2)$, equation (3.11) can be rewritten as

$$S = \frac{\gamma}{2} \int_0^1 t^{n-k+p-1} (1-t^2)^{(k-1)/2} \int_0^1 \frac{z^{(n+p-1)/2}}{(1-z)^{1/2} (1-(1-t^2)z)^{p/2}} dz dt. \quad (3.12)$$

Let us consider the double integral in (3.12). We notice that the first integral can be thought of as an integral over the open interval $(0, 1)$. Hence, we can apply formula [3.197, 3] of [GR94] to the second integral since condition $(1-t^2) < 1$ holds. We get

$$S = \frac{\gamma}{2} B \left(\frac{n+p+1}{2}, \frac{1}{2} \right) \int_0^1 t^{n-k+p-1} (1-t^2)^{(k-1)/2} F \left(\frac{p}{2}, \frac{n+p+1}{2}, \frac{n+p}{2} + 1; 1-t^2 \right) dt.$$

Changing variables by setting $z = 1 - t^2$, we get

$$S = \gamma' \int_0^1 z^{(k-1)/2} (1-z)^{(n-k+p)/2+1} F\left(\frac{p}{2}, \frac{n+p+1}{2}, \frac{n+p}{2} + 1; z\right) dz,$$

where $\gamma' = \gamma/4B((n+p+1)/2, 1/2)$. Applying formula [7.512, 3] we have

$$\begin{aligned} S &= \gamma' \frac{\Gamma((n+p)/2+1) \Gamma((k+1)/2) \Gamma((n+p-k)/2) \Gamma((n-k+1)/2)}{\Gamma((n+p+1)/2) \Gamma(n/2+1) \Gamma((n+p-k+1)/2)} \\ &= \frac{\Gamma((n+p-k)/2) \Gamma((n-k+1)/2)}{\Gamma((n-k)/2) \Gamma((n+p-k+1)/2)}, \end{aligned}$$

which proves the theorem. \square

Theorem 3.2.1 establishes the behavior of the best algorithm for the “worst” positive definite matrix. It is interesting to study the randomized error also for matrices whose eigenvalues are not so badly distributed. The following theorem gives us an upper bound on the randomized error as a function of the difference between the two largest eigenvalues.

Theorem 3.2.2 *For any symmetric positive definite matrix A , let m denote the number of distinct eigenvalues, let r , $r < n - m + 1$ be the multiplicity of the largest eigenvalue λ_1 , and let λ_{r+1} and λ_n be the second largest and the smallest eigenvalue of A . Then for every p , $1 \leq p < +\infty$, we have for $k \geq m$,*

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) = 0$$

and for $k < m$,

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) \leq \begin{cases} f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \left(\frac{\Gamma((r-p)/2) \Gamma((n-r-p)/2)}{\Gamma(r/2) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } p < r \\ f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \ln^{1/p}(f_k(\lambda_1, \lambda_{r+1}, \lambda_n)) \left(\frac{\Gamma(n/2)}{\Gamma(p/2) \Gamma((n-r)/2)} \right)^{1/p} + \beta & \text{for } p = r \\ [f_k(\lambda_1, \lambda_{r+1}, \lambda_n)]^{r/p} \left(\frac{\Gamma(n/2) \Gamma((p-r)/2)}{\Gamma(p/2) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } p > r, \end{cases}$$

where $f_k(\lambda_1, \lambda_{r+1}, \lambda_n)$ is defined in (3.24), and T_k is the Chebyshev polynomial of the first kind of degree $k - 1$.

Before giving the proof, it is worth to comment on this theorem. The first part tells us that, when the number of iterations k reaches the number of distinct eigenvalues, the randomized error vanishes no matter how the eigenvalues are distributed. The second part shows that, when the two largest eigenvalues are well separated, the randomized error quickly decreases with k (in fact $f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \simeq \alpha^{-k}$ for a constant $\alpha > 1$). Moreover, we observe that we have three different behaviors of the randomized error depending on the relation between p and r .

We have not been able to find lower bounds, hence we still do not know if the relation between p and r is intrinsic for the problem or if the three cases above are due to our inability to bound multi-dimensional integrals.

Proof of Theorem 3.2.2. In the next section (see Theorem 3.3.1) we prove that, for $k \geq m$, $e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = 0$. Then, we have

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) \leq e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = 0.$$

We now consider the case $k < m$. We have already observed that

$$\mathbf{u}_k^{\mathcal{B}} = \frac{\sum_{i=1}^n b_i P^*(\lambda_i) \mathbf{z}_i}{\sqrt{\sum_{i=1}^n b_i^2 P^*(\lambda_i)^2}}, \quad (3.13)$$

where P^* is the polynomial in \mathcal{P}_k for which the minimum in (3.4) is achieved. By setting $x_i = \lambda_i/\lambda_1$ and $Q^*(t) = P^*(\lambda_1 t)/P^*(\lambda_1)$, the randomized error can be written as

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) = \left(\int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^*(x_i)^2} \right)^{p/2} \mu(db) \right)^{1/p}. \quad (3.14)$$

Denote by $\mathcal{P}_k(1)$ the class of all polynomials of degree at most $k-1$ that are equal to 1 in 1. Then, by (1.3) we have

$$\begin{aligned} e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) &= \left(\int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^{p/2} \mu(db) \right)^{1/p} \\ &= \left(\int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(1 - \frac{\sum_{i=1}^r b_i^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^{p/2} \mu(db) \right)^{1/p}. \end{aligned} \quad (3.15)$$

Thus, the polynomial Q^* that minimizes the error $e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p)$ is such that

$$\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2 = \min_{Q \in \mathcal{P}_k(1)} \sum_{i=r+1}^n b_i^2 Q^2(x_i).$$

We have

$$\begin{aligned} \min_{Q \in \mathcal{P}_k(1)} \sum_{i=r+1}^n b_i^2 Q^2(x_i) &\leq \min_{Q \in \mathcal{P}_k(1)} \max_{t \in [x_n, x_{r+1}]} \sum_{i=r+1}^n b_i^2 Q^2(t) \\ &= \left[\min_{Q \in \mathcal{P}_k(1)} \max_{t \in [x_n, x_{r+1}]} |Q(t)| \right]^2 \sum_{i=r+1}^n b_i^2. \end{aligned} \quad (3.16)$$

From the Minimax Theorem (see 1.1.1) we have that

$$\min_{Q \in \mathcal{P}_k(1)} \max_{t \in [x_n, x_{r+1}]} |Q(t)| = \frac{1}{\left| T_k \left(1 + 2 \frac{\lambda_1 - \lambda_{r+1}}{\lambda_{r+1} - \lambda_n} \right) \right|},$$

where T_k is the Chebyshev polynomial of the first kind of degree $k-1$.

Let

$$f_k(\lambda_1, \lambda_{r+1}, \lambda_n) = \frac{1}{\left| T_k \left(1 + 2 \frac{\lambda_1 - \lambda_{r+1}}{\lambda_{r+1} - \lambda_n} \right) \right|}.$$

By (3.16) we get

$$\min_{Q \in \mathcal{P}_k(1)} \sum_{i=r+1}^n b_i^2 Q^2(x_i) \leq f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \sum_{i=r+1}^n b_i^2. \quad (3.17)$$

Using (3.17) we upper bound (3.15) as follows

$$[e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p)]^p \leq \int_{\|b\|=1} \left(\frac{f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \sum_{i=r+1}^n b_i^2} \right)^{p/2} db. \quad (3.18)$$

The theorem now follows from the upper bounds proven in Lemma 2.1.2. \square

3.3 Eigenvector Approximation by Lanczos Method

In this section we consider the Lanczos method for eigenvector estimate. In view of Theorem 3.2.1 we know that we cannot have distribution-free bounds. Therefore, we prove a result analogous to Theorem 3.2.2 which bounds the rate of convergence of Lanczos method in terms of the relative distances between the eigenvalues. Note that this theorem shows the substantial equivalence between the Lanczos method and the best polynomial method \mathcal{B} .

Theorem 3.3.1 *Let $\mathbf{u}_k^{\text{Lan}}$ be the vector returned by the Lanczos method after k steps. For any symmetric positive definite matrix A , let m denote the number of distinct eigenvalues, and let r , $r < n - m + 1$, be the multiplicity of the largest eigenvalue λ_1 , and let λ_{r+1} and λ_n be the second largest and the smallest eigenvalue of A . Then, for every p , $1 \leq p < +\infty$, we have for $k \geq m$,*

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = 0$$

and for $k < m$,

$$e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) \leq e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) \leq \sqrt{\frac{\lambda_1 - \lambda_n}{\lambda_1 - \lambda_{r+1}}} e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p).$$

Proof. To prove that $e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = 0$ when $k \geq m$, we use Theorem 5 of [LW96] where this result is proven for $p = 2$. The generalization to the \mathcal{L}_p case is straightforward.

Assume now that $k < m$. Obviously we have $e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p) \leq e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$. Let $\mathbf{b} = \sum_{i=1}^n b_i \mathbf{z}_i$ denote the initial vector. By definition of the Lanczos method we have

$$\mathbf{u}_k^{\text{Lan}} = \frac{\sum_{i=1}^n b_i P^*(\lambda_i) \mathbf{x}_i}{\sqrt{\sum_{i=1}^n b_i^2 P^*(\lambda_i)^2}},$$

where P^* is the polynomial in \mathcal{P}_k which maximizes the Rayleigh quotient. Thus

$$\frac{\sum_{i=1}^n b_i^2 \lambda_i P^*(\lambda_i)^2}{\sum_{i=1}^n b_i^2 P^*(\lambda_i)^2} = \max_{P \in \mathcal{P}_k} \frac{\sum_{i=1}^n b_i^2 \lambda_i P^2(\lambda_i)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)}.$$

Equivalently, P^* minimizes the relative error for the approximation of λ_1 , that is

$$\frac{\sum_{i=r+1}^n b_i^2 P^*(\lambda_i)^2 (1 - \lambda_i/\lambda_1)}{\sum_{i=1}^n b_i^2 P^*(\lambda_i)^2} = \min_{P \in \mathcal{P}_k} \frac{\sum_{i=r+1}^n b_i^2 P^2(\lambda_i) (1 - \lambda_i/\lambda_1)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)}. \quad (3.19)$$

By (1.4), setting $x_i = \lambda_i/\lambda_1$ and $Q^*(t) = P^*(\lambda_1 t)/P^*(\lambda_1)$, we get

$$e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = \left(\int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^*(x_i)^2} \right)^{p/2} \mu(db) \right)^{1/p}. \quad (3.20)$$

We have

$$\begin{aligned}
[e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)]^p &= \int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2 (1-x_i) \frac{1}{(1-x_i)}}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^*(x_i)^2} \right)^{p/2} \mu(db) \\
&\leq \frac{1}{(1-x_{r+1})^{p/2}} \int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^*(x_i)^2 (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^*(x_i)^2} \right)^{p/2} \mu(db) \\
&= \frac{1}{(1-x_{r+1})^{p/2}} \int_{\|b\|=1} \left(\min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i) (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^{p/2} \mu(db), \quad (3.21)
\end{aligned}$$

where (3.21) follows from 3.19 by setting $Q(t) = P(\lambda_1 t)/P(\lambda_1)$. Since

$$\min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i) (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \leq (1-x_n) \min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)},$$

using (3.15) we get

$$\begin{aligned}
e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) &\leq \left(\frac{1-x_n}{1-x_{r+1}} \right)^{1/2} \left(\int_{\|b\|=1} \left(\min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^{p/2} \mu(db) \right)^{1/p} \\
&= \sqrt{\frac{\lambda_1 - \lambda_n}{\lambda_1 - \lambda_{r+1}}} e^{\text{ran}}(\mathbf{u}_k^{\mathcal{B}}, A, p).
\end{aligned}$$

□

The randomized error $e^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$ has been studied in [LW96] for $p = 2$. Since we take into account the multiplicity r of λ_1 , we obtain tighter estimates. For $r \geq 2$, we get asymptotically (with respect to k) better bounds; for $r = 1$, we get the same asymptotic behavior but with a smaller constant.

3.4 Eigenvalue Approximation by The Lanczos Method

We already underlined in Chapter 1 that the Lanczos method is the method, among polynomial algorithms, that better approximates the largest eigenvalue λ_1 . In fact the value ξ_k^{Lan} returned by the method maximizes the Rayleigh quotient over the space \mathcal{K}_k .

By equation (1.2), the randomized error in the \mathcal{L}_p sense for eigenvalue estimate by the Lanczos method is given by

$$\begin{aligned}
e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) &= \left(\int_{S_n} \left(\min_{P \in \mathcal{P}_k} \frac{\sum_{i=r+1}^n b_i^2 P^2(\lambda_i) (1-\lambda_i/\lambda_1)}{\sum_{i=1}^n b_i^2 P^2(\lambda_i)} \right)^p \mu(db) \right)^{1/p} \\
&= \left(\int_{S_n} \left(\min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i) (1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^p \mu(db) \right)^{1/p}. \quad (3.22)
\end{aligned}$$

As for the power method, also for the Lanczos method, it is possible to bound $e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$ (see Section (3.4.1)) by a quantity that does not depend on the eigenvalues of the matrix.

3.4.1 Distribution free bounds

In this section we first report a theorem due to Kuczyński and Woźniakowski that gives a bound of the randomized error $e^{\text{ran}}(\xi_k^{\text{Lan}}, A, 1)$ that is independent of the distribution of the eigenvalues but that holds only for the \mathcal{L}_1 case. In Theorem 3.4.2 we generalize this result to the \mathcal{L}_p case when the multiplicity of the largest eigenvalue is known. Unfortunately, for the Lanczos method, is not very easy to get lower bound, so we can only judge the goodness of our upper bound by comparing the theoretical results with the values obtained from tests (see Section 3.5).

Theorem 3.4.1 [KW92a] *Let ξ_k^{Lan} be the value returned by the Lanczos method applied to the matrix A after k steps. Let m denote the number of distinct eigenvalues of A . Then for $k \geq m$*

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, 1) = 0,$$

and, for $k < m$,

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, 1) \leq 0.103 \left(\frac{\ln(n(k-1)^4)}{k-1} \right)^2 \leq 2.575 \left(\frac{\ln n}{k-1} \right)^2.$$

Proof. We refer to the paper of Kuczyński and Woźniakowski [KW92a] for the proof of this theorem.

□

Next theorem gives a distribution-free bound when the \mathcal{L}_p norm is considered. We note that we have three different upper bounds depending on the relation between the multiplicity r of λ_1 and the norm parameter p . The lack of lower bound does not allow us to find out if the Lanczos method follows these different behaviors. We can only compare these bounds with the numerical bounds (see Section 3.5).

Theorem 3.4.2 *Let ξ_k^{Lan} be the value returned by the Lanczos method applied to the matrix A after k steps. Let m denote the number of distinct eigenvalues of A , let r , $r \leq n - m + 1$, be the multiplicity of the largest eigenvalue λ_1 . For any $p \in [1, +\infty)$ we have*
for $k \geq m$

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) = 0,$$

and, for $k < m$,

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) \leq \begin{cases} \frac{1}{k^2} \left(\frac{\Gamma((r-2p)/2) \Gamma((n-r+2p)/2)}{\Gamma(r/2) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } 2p < r \\ \frac{1}{k^2} \ln^{1/p}(k+1)^2 \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-2p)/2)} \right)^{1/p} + \beta & \text{for } 2p = r \\ \frac{1}{k^{(r/p)}} \left(\frac{\Gamma((2p-r)/2) \Gamma(n/2)}{\Gamma(p) \Gamma((n-r)/2)} \right)^{1/p} & \text{for } 2p > r \end{cases}$$

where

$$\beta = \frac{1}{k^2} \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-2p)/2)} \left(2 + \frac{2}{n} \right) \right)^{1/p}.$$

Proof. To prove that $e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) = 0$ when $k \geq m$, we use Theorem 3.4.1 where this result is proven for $p = 1$. The generalization to the \mathcal{L}_p case is straightforward.

Assume now $k < m$. Accordingly with (3.22), the randomized error is given by

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) = \left(\int_{\|b\|=1} \left(\min_{Q \in \mathcal{P}_k(1)} \frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^p db \right)^{1/p}.$$

Rising everything to the power p , we get

$$\begin{aligned} [e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p &= \int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^p \mu(db) \\ &\leq \int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i^2)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)} \right)^p \mu(db). \end{aligned}$$

Since $\sum_{i=r+1}^n b_i^2 Q^2(x_i) \geq \sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i^2)$, from the previous equation we get

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq \int_{\|b\|=1} \min_{Q \in \mathcal{P}_k(1)} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i^2)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i^2)} \right)^p \mu(db). \quad (3.23)$$

Let U_k be the Chebyshev polynomial of second kind of degree $k-1$. The polynomial $W_k = U_k/k$ is in the class $\mathcal{P}_k(1)$. We replace the minimum in (3.23) with the value achieved for W_k , and we get

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq \int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 W_k^2(x_i)(1-x_i^2)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 W_k^2(x_i)(1-x_i^2)} \right)^p \mu(db).$$

Rewriting last equation as

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq \int_{\|b\|=1} \left(1 - \frac{\sum_{i=1}^r b_i^2}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 W_k^2(x_i)(1-x_i^2)} \right)^p \mu(db),$$

we notice that we can upper bound $e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$ by taking the maximum of $W_k^2(x)(1-x^2)$ over $[0, 1]$. By the properties of U_k , we have

$$\begin{aligned} \max_{x \in [0,1]} W_k^2(x)(1-x^2) &= \max_{x \in [0,1]} \frac{U_k^2(x)}{k^2} (1-x^2) \\ &= \max_{\theta \in [0, \pi/2]} \frac{U_k^2(\cos(\theta))}{k^2} (1-\cos^2(\theta)) \\ &= \max_{\theta \in [0, \pi/2]} \frac{\sin^2(k\theta) \sin^2(\theta)}{\sin^2(\theta) k^2} = \frac{1}{k^2}. \end{aligned}$$

We get

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq \int_{\|b\|=1} \left(\frac{(1/k^2) \sum_{i=r+1}^n b_i^2}{\sum_{i=1}^r b_i^2 + (1/k^2) \sum_{i=r+1}^n b_i^2} \right)^p \mu(db).$$

The theorem now follows from the upper bounds proven in lemma 2.1.2 \square

Theorem 3.4.2 establishes upper bounds that hold for the \mathcal{L}_p norm. For these bounds considerations similar to those for the power method apply. That is, for $p = 1$ these bounds are tighter than those reported in [KW92a] only when the multiplicity of the largest eigenvalue is greater than $n/\ln^2 n$.

3.4.2 Distribution Dependent Bounds

We now give upper bounds on the randomized error for approximation of the largest eigenvalue by the Lanczos method. The same problem has been considered for the \mathcal{L}_1 norm in [KW92a]. By comparing our results (for $p = 1$) with those reported in [KW92a] we find that our estimates are tighter. For $r \geq 2$, we get asymptotically (with respect to k) better bounds; for $r = 1$, we get the same asymptotic behavior but with a smaller constant.

Theorem 3.4.3 *For any symmetric positive definite matrix A , let m be the number of distinct eigenvalues of A . Let $r, r < n - m + 1$, denote the multiplicity of the largest eigenvalue λ_1 , and let λ_{r+1} and λ_n be the second largest and the smallest eigenvalue of A . For every $p \in [1, +\infty)$ and $k < m$ we have*

$$e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) \leq \begin{cases} \left(1 - \frac{\lambda_n}{\lambda_1}\right) (f_k(\lambda_1, \lambda_{r+1}, \lambda_n))^2 \left(\frac{\Gamma((n+2p-r)/2) \Gamma((r-2p)/2)}{\Gamma((n-r)/2) \Gamma(r/2)}\right)^{1/p} & \text{for } 2p < r, \\ \left(1 - \frac{\lambda_n}{\lambda_1}\right) (f_k(\lambda_1, \lambda_{r+1}, \lambda_n))^2 \left(\frac{2 \ln(f_k(\lambda_1, \lambda_{r+1}, \lambda_n)) \Gamma(n/2)}{\Gamma((n-r)/2) \Gamma(p)}\right)^{1/p} + \beta & \text{for } 2p = r, \\ \left(1 - \frac{\lambda_n}{\lambda_1}\right) (f_k(\lambda_1, \lambda_{r+1}, \lambda_n))^{r/p} \left(\frac{\Gamma(n/2) \Gamma((2p-r)/2)}{\Gamma((n-r)/2) \Gamma(p)}\right)^{1/p} & \text{for } 2p > r, \end{cases}$$

where

$$\beta = (f_k(\lambda_1, \lambda_{r+1}, \lambda_n))^2 \left(\frac{\Gamma(n/2)}{\Gamma(p) \Gamma((n-2p)/2)} \left(2 + \frac{2}{n}\right)\right)^{1/p},$$

and

$$f_k(\lambda_1, \lambda_{r+1}, \lambda_n) = \frac{1}{\left|T_k\left(1 + 2 \frac{\lambda_1 - \lambda_{r+1}}{\lambda_{r+1} - \lambda_n}\right)\right|} \quad \beta = f_k(\lambda_1, \lambda_{r+1}, \lambda_n) \left(\frac{\Gamma(n/2)}{\Gamma(p/2) \Gamma((n-r)/2)} \left(2 + \frac{2}{n}\right)\right)^{1/p}. \quad (3.24)$$

Proof. By equation (3.23) we know that

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq \min_{Q \in \mathcal{P}_k(1)} \int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)(1-x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)}\right)^p \mu(db).$$

Since $x_{r+1} \geq x_{r+2} \geq \dots \geq x_n$, we get

$$[e^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)]^p \leq (1 - x_n)^p \min_{Q \in \mathcal{P}_k(1)} \int_{\|b\|=1} \left(\frac{\sum_{i=r+1}^n b_i^2 Q^2(x_i)}{\sum_{i=1}^r b_i^2 + \sum_{i=r+1}^n b_i^2 Q^2(x_i)}\right)^p \mu(db). \quad (3.25)$$

To complete the proof we note that the integral at the right hand side has been already bounded in Theorem 3.2.2. □

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	p
10	$6.724e - 03$	$1.151e + 01$	$9.240e - 01$	$3.427e + 02$	1
30	$8.158e - 04$	$6.373e + 00$	$7.273e - 01$	$2.397e + 02$	1
90	$4.427e - 05$	$4.766e - 01$	$1.180e - 01$	$3.479e + 01$	1
10	$6.874e - 03$	$2.398e + 00$	$9.209e - 01$	$3.999e + 02$	2
30	$9.444e - 04$	$1.785e + 00$	$7.639e - 01$	$2.946e + 02$	2
90	$1.064e - 04$	$4.881e - 01$	$2.129e - 01$	$8.032e + 01$	2
10	$9.725e - 03$	$1.079e + 00$	$9.528e - 01$	$2.843e + 02$	10
30	$1.945e - 03$	$1.017e + 00$	$8.858e - 01$	$2.674e + 02$	10
90	$3.595e - 04$	$7.845e - 01$	$7.287e - 01$	$2.062e + 02$	10

Table 3.1: Lanczos method: Chebyshev distribution, where $r = 1$.

3.5 Numerical Experiments

This section reports the results of some numerical experiments designed to verify the sharpness of the bounds provided in this thesis. We report tests for eigenvalue and eigenvector approximation by the Lanczos method. All tests have been performed on a Silicon Graphics Iris Indigo R4000 using double precision. The Lanczos method has been implemented using the numerical package *Meschach* [LS94].

We tested the Lanczos method using matrices of size 100 with eigenvalues distributed as in Section 2.6.

In Section 2.6 we already pointed out that, without loss of generality, we can restrict ourselves to consider diagonal matrices. For each matrix we generated 100 starting vectors uniformly distributed over the unit sphere using the technique described in Section 1.1. Estimates of the randomized errors $\varepsilon^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$ and $\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$, have been computed by the average over the random starting vectors, i.e.

$$\varepsilon^{\text{ran}}(\xi_k^{\text{Lan}}, A, p) = \left(\frac{1}{100} \sum_{i=1}^{100} \left(\frac{\lambda_1 - \xi_k^{\text{Lan}}}{\lambda_1} \right)^p \right)^{1/p}, \quad (3.26)$$

$$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p) = \left(\frac{1}{100} \sum_{i=1}^{100} \min_{\mathbf{v} \in \mathbf{Z}_1} \|\mathbf{u}_k^{\text{Lan}} - \mathbf{v}\|^p \right)^{1/p}. \quad (3.27)$$

Let $\varepsilon^{\text{ub}}(\xi_k^{\text{Lan}}, A, p)$ and $\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Lan}}, A, p)$ denote the upper bounds for Lanczos method given by Theorems 3.4.3 and 3.3.1 respectively. Finally, k denotes the number of iterations and p the norm parameter.

The results of the numerical tests are reported in Tables 3.1–3.3. For reasons of space we cannot report the numerical results for all distributions. Since no lower bounds are known for Lanczos method, in the randomized setting, these tests are particularly important for establishing the sharpness of the upper bounds given by Theorems 3.4.3 and 3.3.1. Table 3.1 shows the results for the Chebyshev distribution, which is the one with the slowest convergence rate. Table 3.2 shows the results for the exponential distribution 1, for which the error reaches the machine precision (10^{-16}) after only 25 iterations.

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	p
10	$5.132e - 05$	$1.765e - 03$	$9.964e - 04$	$4.920e - 03$	1
15	$1.410e - 12$	$8.878e - 06$	$1.089e - 07$	$3.573e - 05$	1
25	$5.432e - 16$	$2.245e - 10$	$3.019e - 15$	$1.459e - 09$	1
10	$9.619e - 04$	$2.932e - 02$	$9.481e - 03$	$1.367e - 01$	2
15	$3.837e - 11$	$2.079e - 03$	$1.378e - 06$	$9.693e - 03$	2
25	$6.161e - 16$	$1.045e - 05$	$1.255e - 14$	$4.874e - 05$	2
10	$1.372e - 02$	$4.376e - 01$	$1.107e - 01$	$4.890e + 00$	10
15	$6.052e - 10$	$2.578e - 01$	$2.016e - 05$	$2.880e + 00$	10
25	$1.393e - 15$	$8.945e - 02$	$1.744e - 13$	$9.994e - 01$	10

Table 3.2: Lanczos method: Exponential Distribution 1, where $r = 1$.

k	$\varepsilon^{\text{ran}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\xi_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ran}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	$\varepsilon^{\text{ub}}(\mathbf{u}_k^{\text{Lan}}, A, p)$	p
10	$1.238e - 03$	$1.628e + 00$	$4.060e - 01$	$2.368e + 01$	1
30	$4.410e - 06$	$9.006e - 03$	$2.967e - 02$	$1.728e + 00$	1
60	$1.202e - 12$	$2.874e - 06$	$7.305e - 06$	$3.086e - 02$	1
10	$1.820e - 03$	$7.877e + 00$	$4.817e - 01$	$2.770e + 01$	5
30	$1.573e - 05$	$5.180e - 02$	$4.631e - 02$	$2.021e + 00$	5
60	$3.428e - 12$	$1.816e - 05$	$1.016e - 05$	$3.609e - 02$	5
10	$2.390e - 03$	$6.903e - 01$	$5.603e - 01$	$7.432e + 01$	10
30	$2.259e - 05$	$5.135e - 02$	$6.035e - 02$	$5.850e + 00$	10
60	$8.472e - 12$	$9.172e - 04$	$1.709e - 05$	$1.089e - 01$	10

Table 3.3: Lanczos method: Logarithmic Distribution, where $r = 10$.

We can observe, especially from Table 3.2, that the speed of convergence shows a dependence on the norm parameter. In fact, we have slowest convergence as the value of p increases. Unfortunately, the upper bounds are, in this case, not very accurate, so it is not easy to understand if the dependence on p and r is really like that of Theorems 3.4.3 and 3.3.1.

In Table 3.3 we report the numerical results obtained for the Logarithmic distribution when the multiplicity of the largest eigenvalue is 10.

Chapter 4

A Randomized Parallel Algorithm for Eigenvalue Approximation

The algorithms presented in the first part of this thesis can be considered as randomized methods even though the randomization is only limited to the choice of the initial vector, and no random choice is done in the other parts of the algorithms. In this chapter we present a “fully” randomized version of the power method. The parallel cost of this algorithm is lower than that of the best deterministic implementation of the power method on a PRAM.

In the area of numerical linear algebra there are very few examples of randomized algorithms. An interesting example is provided by the result obtained by Codenotti and Leoncini [CL91]. They propose a parallel Monte Carlo method for solving linear systems. The computational parallel cost of their algorithm is $O(\log n)$ with $n^{O(1)}$ processors, while the best known deterministic algorithm for this problem takes $O(\log^2 n)$ time with $n^{O(1)}$ processors.

The results of this chapter are also reported in [dCo95].

4.1 The Algorithm

Let A be an $n \times n$ symmetric real matrix, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the eigenvalues of A . Assume

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|,$$

and that if $|\lambda_1| = |\lambda_2| = \dots = |\lambda_r|$ then we have $\lambda_1 = \lambda_2 = \dots = \lambda_r$, that is A does not have two different dominant eigenvalues. Denote by $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ the orthogonal eigenvectors of A .

The randomized method we are proposing is an algorithm based on the power method without normalization, that can be defined as follows

$$\begin{cases} \mathbf{u}^{(0)} &= \mathbf{b}, \\ \mathbf{u}^{(k)} &= A\mathbf{u}^{(k-1)}, \end{cases} \quad (4.1)$$

where \mathbf{b} is a starting vector. If the starting vector \mathbf{b} is not orthogonal to the eigenspace corresponding to the dominant eigenvalue λ_1 , then the sequence $\mathbf{u}^{(k)}$ is such that

$$\lim_{k \rightarrow +\infty} \frac{u_m^{(k)}}{u_m^{(k-1)}} = \lambda_1,$$

where $u_m^{(i)}$ denotes the m -th entry of the vector $\mathbf{u}^{(i)}$ and m is an index for which $u_m^{(k-1)} \neq 0$. We can write \mathbf{b} as a linear combination of the orthogonal eigenvectors. $\mathbf{b} = \sum_{i=1}^n b_i \mathbf{z}_i$. Assuming \mathbf{b} is not orthogonal to \mathbf{Z}_1 , then $\sum_{i=1}^r b_i \neq 0$. Substituting into (4.1) we get

$$\frac{u_m^{(k)}}{u_m^{(k-1)}} = \frac{\sum_{i=1}^n b_i \lambda_i^k z_m^{(i)}}{\sum_{i=1}^n b_i \lambda_i^{k-1} z_m^{(i)}} = \lambda_1 \frac{\sum_{i=1}^r b_i z_m^{(i)} + \sum_{i=r+1}^n b_i \left(\frac{\lambda_i}{\lambda_1}\right)^k z_m^{(i)}}{\sum_{i=1}^r b_i z_m^{(i)} + \sum_{i=r+1}^n b_i \left(\frac{\lambda_i}{\lambda_1}\right)^{k-1} z_m^{(i)}}. \quad (4.2)$$

Since $(\lambda_i/\lambda_1)^k$ goes to zero as k goes to infinity, the ratio $u_m^{(k)}/u_m^{(k-1)}$ converges to λ_1 .

In this chapter $\lambda_1^{(A)}$ (or simply λ_1 when there is not ambiguity) denotes the eigenvalue of the matrix A whose absolute value is maximum.

The deterministic computation of an approximation to λ_1 with this scheme, involves the calculation of the vectors $\mathbf{u}^{(k)} = A^k \mathbf{b}$ and $\mathbf{u}^{(k-1)} = A^{k-1} \mathbf{b}$, for a sufficiently large k . On a CREW-PRAM this takes $O(\log k \log n)$ time with $n^3/\log n$ processors as a consequence of the computation of the k -th power of the matrix A . We assume the starting vector \mathbf{b} is not orthogonal to the eigenspace \mathbf{Z}_1 . We show a randomized method for the computation of the two quantities $u_m^{(k)}$ and $u_m^{(k-1)}$, that allows us to estimate λ_1 as the ratio between these two values.

The randomized method for determining an approximation of $u_m^{(k)}$ consists in a n -fold sampling of the value of the random variable with a series of independent tests and in the computation of their arithmetic mean. The m -th entry of $\mathbf{u}^{(k)} = A^k \mathbf{b}$ can be written as

$$u_m^{(k)} = \sum_{i_1, i_2, \dots, i_k} a_{mi_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} b_{i_k}. \quad (4.3)$$

Assume that

$$\begin{aligned} a_{ij} &= f_{ij} p_{ij}, & 0 < p_{ij} < 1, & & i, j &= 1, 2, \dots, n \\ \sum_{j=1}^n p_{ij} &= 1, & i &= 1, 2, \dots, n, \end{aligned} \quad (4.4)$$

then, (4.3) can be rewritten as

$$u_m^{(k)} = \sum_{i_1, i_2, \dots, i_k} f_{mi_1} f_{i_1 i_2} \cdots f_{i_{k-1} i_k} p_{mi_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} b_{i_k}.$$

Let P be an $n \times n$ matrix with entries p_{ij} . By (4.4), we notice that P is a transition matrix that describes a family of homogeneous Markov chains with state space $V = 1, 2, \dots, n$ (see [Fel68] for a theoretical treatment of Markov chains).

Let X_0, X_1, \dots, X_k be the Markov chain of length $k+1$ described by P . If we concentrate the initial distribution in m , i.e., $\Pr(X_0 = m) = 1$, we have

$$\Pr(X_0 = m, X_1 = i_1, \dots, X_k = i_k) = p_{mi_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k}. \quad (4.5)$$

Let the $(k + 1)$ -tuple m, i_1, i_2, \dots, i_k denote a trajectory of the Markov process. Let Ω be the set of all trajectories of length $k + 1$ with m as initial state. We construct the random variable $\Phi_m^{(k)} : \Omega \rightarrow \mathbb{R}$ as

$$\Phi_m^{(k)}(m, i_1, i_2, \dots, i_k) = f_{mi_1} f_{i_1 i_2} \cdots f_{i_{k-1} i_k} q_{i_k}. \quad (4.6)$$

The expectation of $\Phi_m^{(k)}$ is equal to $u_m^{(k)}$. In fact, by definition of expectation, together with (4.5) and (4.6), we have

$$\begin{aligned} E(\Phi_m^{(k)}) &= \sum_{i_1, i_2, \dots, i_k} \Phi_m^{(k)}(m, i_1, i_2, \dots, i_k) \Pr(m, i_1, i_2, \dots, i_k) \\ &= \sum_{i_1, i_2, \dots, i_k} f_{mi_1} f_{i_1 i_2} \cdots f_{i_{k-1} i_k} p_{mi_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} q_{i_k} = u_m^{(k)}. \end{aligned}$$

For approximating the expected value of $\Phi_m^{(k)}$, we proceed as follows

1. Compute N independent trajectories $\omega_1, \omega_2, \dots, \omega_N$.
2. Compute $\tilde{u}_m^{(k)} = \sum_{j=1}^N \Phi_m^{(k)}(\omega_j) / N$.

For a sufficiently large N , the quantity $\tilde{u}_m^{(k)}$ is close to $u_m^{(k)}$. The Chebyshev inequality allows us to bound the probabilistic error. We have

$$\begin{aligned} \Pr\left(\left|\frac{u_m^{(k)} - \tilde{u}_m^{(k)}}{u_m^{(k)}}\right| > \varepsilon\right) &= \\ &= \Pr\left(\left|u_m^{(k)} - \tilde{u}_m^{(k)}\right| > \varepsilon |u_m^{(k)}|\right) \leq \frac{\text{Var}\left(\Phi_m^{(k)}\right)}{N(\varepsilon |u_m^{(k)}|)^2}. \end{aligned} \quad (4.7)$$

A problem arising with inequality (4.7) is that, for an arbitrary matrix A , it is difficult to bound the variance of $\Phi_m^{(k)}$. Moreover, it is not possible to give bounds on $|u_m^{(k)}|$ that holds for all the class of symmetric matrices. We circumvent these problems by introducing a matrix M for which these two quantities can be bounded. The new matrix M has the form

$$M = \frac{A}{4k\|A\|_\infty} + I, \quad (4.8)$$

where k is a fixed integer.

We apply the previous method to the matrix M in order to get an approximation to the dominant eigenvalue of M and then we get $\lambda_1^{(A)}$ by the following relation

$$\lambda_i^{(M)} = \frac{\lambda_i^{(A)}}{4k\|A\|_\infty} + 1, \quad i = 1, 2, \dots, n. \quad (4.9)$$

It is evident that the largest eigenvalue of M might not correspond to the largest eigenvalue of A , this happens for example if $\lambda_1^{(A)} < 0$, In this case, it is necessary to apply the algorithm to the matrix

$$\bar{M} = -A/(4k\|A\|_\infty) + I$$

rather than to M . If we do not know the sign of $\lambda_1^{(A)}$, we can compute both $\lambda_1^{(M)}$ and $\lambda_1^{(\bar{M})}$. The dominant eigenvalue of A is $4k\|A\|_\infty(\lambda_1^{(M)} - 1)$ or $4k\|A\|_\infty(\lambda_1^{(\bar{M})} - 1)$ depending on which of those two values has maximum absolute value.

In order to approximate the values $\lambda_1^{(M)}$ and $\lambda_1^{(\bar{M})}$, we proceed according to the procedure shown before. In particular we need to compute the m -th entries of $\mathbf{u}^{(k)} = M^k \mathbf{b}$ and $\mathbf{u}^{(k-1)} = M^{k-1} \mathbf{b}$ (and the corresponding values for \bar{M}).

As before, we decompose each entry m_{ij} of M as the product of f_{ij} and p_{ij} , where

$$f_{ij} = \text{sign}(m_{ij}) \sum_{h=1}^n |m_{ih}|, \quad p_{ij} = \frac{|m_{ij}|}{\sum_{h=1}^n |m_{ih}|}.$$

This choice satisfies conditions (4.4), so that we can implement the Markovian process on the random variable $\Phi_m^{(k)}$.

In the following is sketched the parallel algorithm.

Algorithm RANPOWER

1. Apply the following procedure two times, one when $C = M$, the other when $C = \bar{M}$ to produce the values $\tilde{\lambda}_M^{(k)}$ and $\tilde{\lambda}_{\bar{M}}^{(k)}$ as approximations to the dominant eigenvalues of M and \bar{M} :

- 1.1 Compute $f_{ij} = \text{sign}(c_{ij}) \sum_{h=1}^n |c_{ih}|$ and $p_{ij} = \frac{|c_{ij}|}{|f_{ij}|}$ for $i, j = 1, 2, \dots, n$.
- 1.2 Compute the matrix $S = (s_{ij})$, where $s_{ij} = \sum_{h=1}^j p_{ih}$ for $i, j = 1, 2, \dots, n$.
- 1.3 Compute the trajectories $\omega_1, \dots, \omega_N$, and ν_1, \dots, ν_N as follows:
for each pair (ω_i, ν_i)

1.3.1. Produce, independently, numbers x_{ij} for $i = 1, \dots, n$, $j = 1, \dots, k$, and y_{ij} for $i = 1, \dots, n$, $j = 1, \dots, k-1$, randomly chosen in $[0, 1]$.

1.3.2. Determine the trajectories $\omega_i = t_0, t_1, \dots, t_k$, and $\nu_i = u_0, u_1, \dots, u_{k-1}$ as follows:
Determine integers α_{ij} and β_{ir} for $i = 1, \dots, n$, $j = 1, \dots, k$, and $r = 1, \dots, k-1$, for which

$$s_{i\alpha_{ij-1}} \leq x_{ij} \leq s_{i\alpha_{ij}}, \quad s_{i\beta_{ir-1}} \leq y_{ir} \leq s_{i\beta_{ir}}.$$

The trajectories are

$$\begin{cases} t_0 = m; \\ t_i = \alpha_{t_{i-1}, i} \end{cases} \quad i = 1, \dots, k \quad \begin{cases} u_0 = m; \\ u_r = \beta_{u_{r-1}, r} \end{cases} \quad r = 1, \dots, k-1.$$

- 1.4 Compute the values $\Phi_m^{(k)}(\omega_i)$, and $\Phi_m^{(k-1)}(\nu_i)$ for $i = 1, \dots, N$.
- 1.5 Compute the sums $\sum_{i=1}^N \Phi_m^{(k)}(\omega_i)$ and $\sum_{i=1}^N \Phi_m^{(k-1)}(\nu_i)$.
- 1.6 Compute the approximate value to $\lambda_C^{(k)}$ as

$$\tilde{\lambda}_C^{(k)} = \frac{y_m^{(k)}}{y_m^{(k-1)}} = \frac{\sum_{i=1}^N \Phi_m^{(k)}(\omega_i)}{\sum_{i=1}^N \Phi_m^{(k-1)}(\nu_i)}.$$

2. Compute the values $\mu = 4k \|A\|_\infty (\tilde{\lambda}_M^{(k)} - 1)$, and $\mu' = -4k \|A\|_\infty (\tilde{\lambda}_{\bar{M}}^{(k)} + 1)$.

3. Return the approximation of $\lambda_A^{(k)}$ as

$$\tilde{\lambda}_A^{(k)} = \begin{cases} \mu & \text{if } |\mu| = \max\{|\mu|, |\mu'|\}, \\ \mu' & \text{otherwise.} \end{cases}$$

4.2 Error Analysis

In this section we analyze the error in the computation of an estimate of $\lambda_1^{(A)}$ when we use algorithm RANPOWER.

From Section 4.1 it is clear that the error analysis has to take into account three sources of error:

1. the error arising from the approximation of $u_m^{(k)}/u_m^{(k-1)}$ with the output $\tilde{\lambda}_A^{(k)}$ of the randomized algorithm RANPOWER (probabilistic error);
2. the error due to the truncation of the sequence (4.1) at the k -th term (analytic error);
3. the roundoff error induced by using formula (4.9) to compute an estimate of $\lambda_1^{(A)}$.

Let $\lambda_A^{(k)}$ be the estimate of $\lambda_1^{(A)}$ after k iterations of the power method and let $\tilde{\lambda}_A^{(k)}$ be the value returned by the algorithm RANPOWER.

We would like to determine $\tilde{\lambda}_A^{(k)}$, so that

$$\frac{|\lambda_1^{(A)} - \tilde{\lambda}_A^{(k)}|}{|\lambda_1^{(A)}|} \leq \varepsilon_A, \quad (4.10)$$

where ε_A is a fixed positive threshold.

Observe that, neglecting second order terms, it is sufficient to have

$$\frac{|\lambda_1^{(A)} - \lambda_A^{(k)}|}{|\lambda_1^{(A)}|} \leq \frac{1}{2}\varepsilon_A, \quad (4.11)$$

$$\frac{|\lambda_A^{(k)} - \tilde{\lambda}_A^{(k)}|}{|\lambda_A^{(k)}|} \leq \frac{1}{2}\varepsilon_A. \quad (4.12)$$

In fact, from (4.11) we have

$$-\frac{1}{2}\varepsilon_A|\lambda_1^{(A)}| \leq \lambda_1^{(A)} - \lambda_A^{(k)} \leq \frac{1}{2}\varepsilon_A|\lambda_1^{(A)}|$$

and from (4.12)

$$-\frac{1}{2}\varepsilon_A|\lambda_A^{(k)}| \leq \lambda_A^{(k)} - \tilde{\lambda}_A^{(k)} \leq \frac{1}{2}\varepsilon_A|\lambda_A^{(k)}|.$$

Summing up these two inequalities we get

$$-\varepsilon_A|\lambda_1^{(A)}| - \frac{1}{2}\varepsilon_A(|\lambda_A^{(k)}| - |\lambda_1^{(A)}|) \leq \lambda_1^{(A)} - \tilde{\lambda}_A^{(k)} \leq \varepsilon_A|\lambda_1^{(A)}| + \frac{1}{2}\varepsilon_A(|\lambda_A^{(k)}| - |\lambda_1^{(A)}|).$$

Since $|a| - |b| \leq |a - b|$, and we have

$$\frac{|\lambda_1^{(A)} - \tilde{\lambda}_A^{(k)}|}{|\lambda_1^{(A)}|} \leq \varepsilon_A + \frac{1}{4}\varepsilon_A^2.$$

The analysis of the analytic error (4.11) depends on specific properties of the matrix A . In particular, from (4.2) we have

$$\frac{|\lambda_1^{(A)} - \lambda_A^{(k)}|}{|\lambda_1^{(A)}|} = \left| \frac{\sum_{i=r+1}^n b_i \left(\frac{\lambda_i^{(A)}}{\lambda_1^{(A)}}\right)^{k-1} z_m^{(i)} \left(1 - \frac{\lambda_i^{(A)}}{\lambda_1^{(A)}}\right)}{\sum_{i=1}^r b_i z_m^{(i)} + \sum_{i=r+1}^n b_i \left(\frac{\lambda_i^{(A)}}{\lambda_1^{(A)}}\right)^{k-1} z_m^{(i)}} \right|$$

$$\begin{aligned}
&\leq \left| \frac{\left(\frac{\lambda_{r+1}^{(A)}}{\lambda_1^{(A)}}\right)^{k-1} \sum_{i=r+1}^n b_i z_m^{(i)}}{\sum_{i=1}^r b_i z_m^{(i)} + \left(\frac{\lambda_{r+1}^{(A)}}{\lambda_1^{(A)}}\right)^{k-1} \sum_{i=r+1}^n b_i z_m^{(i)}} \right| \\
&\doteq \left(\frac{|\lambda_{r+1}^{(A)}|}{|\lambda_1^{(A)}|} \right)^{k-1} \left| \frac{\sum_{i=r+1}^n b_i z_m^{(i)}}{\sum_{i=1}^r b_i z_m^{(1)}} \right|,
\end{aligned}$$

from which follows that, in order to satisfy (4.11), it is sufficient to choose

$$k = O(\log(\varepsilon_A) / \log(|\lambda_{r+1}| / |\lambda_1|)).$$

Most of the time we do not have a good approximation of λ_{r+1}/λ_1 , since we are looking for a good estimate of λ_1 . Moreover, inequality (4.11) is not sufficient to guarantee that a similar bound on the analytic error holds also when we apply the method to the matrix M . This problem can be partially resolved by the application of the algorithm RANPOWER many times, with increasing values of k . We stop this process when the difference between two successive approximation of $\lambda_1^{(M)}$ is less than a positive threshold, say ε_M . It is well known that this condition does not guarantee the same bound on the actual error but for iterative method it is necessary to establish some halting criterion.

We now consider the roundoff error. We assume that k is a fixed integer value corresponding to the number of iterations that guarantees bound (4.11).

The first step of the algorithm RANPOWER consists in the computation of the estimate of the largest eigenvalues of M and \bar{M} . Let us denote by $\tilde{\lambda}_A^{(k)}$ the approximation of $\lambda_1^{(A)}$ returned by RANPOWER and by $\tilde{\lambda}_M^{(k)}$ and $\tilde{\lambda}_{\bar{M}}^{(k)}$ respectively the estimate of largest eigenvalues of M and \bar{M} computed by RANPOWER.

We assume $\lambda_1^{(A)} \geq 0$, the case $\lambda_1(A) \leq 0$ is similar. With this assumption, we have that at step **3**. of the algorithm RANPOWER the algorithm outputs $\mu = 4k\|A\|_\infty (\tilde{\lambda}_M^{(k)} - 1)$ as approximation of $\lambda_A^{(k)}$. We need then to analyze how the probabilistic error, on $\tilde{\lambda}_M^{(k)}$ and on $\tilde{\lambda}_{\bar{M}}^{(k)}$, is propagated to $\tilde{\lambda}_A^{(k)}$ through relation (4.9).

Assume that $\frac{|\lambda_M^{(k)} - \tilde{\lambda}_M^{(k)}|}{|\lambda_M^{(k)}|} \leq \frac{1}{2}\varepsilon_M$, for a certain positive threshold ε_M that depends on the randomized error. Assume also that k and the entries of the matrix A are numbers exactly available. Suppose to compute λ_1 with the following procedure:

$$\begin{aligned}
\mathbf{y}^{(1)} &= \|A\|_\infty \\
\mathbf{y}^{(2)} &= 4 * k \\
\mathbf{y}^{(3)} &= \mathbf{y}^{(2)} * \mathbf{y}^{(1)} \\
\mathbf{y}^{(4)} &= \lambda_M^{(k)} - 1 \\
\mathbf{y}^{(5)} &= \mathbf{y}^{(3)} * \mathbf{y}^{(4)}.
\end{aligned} \tag{4.13}$$

Let $\varepsilon_t^{(i)}$ be the error at steps (i) in (4.13). Then the total error ε_t in the computation of $\lambda_1^{(A)}$ with this procedure is denoted by $\varepsilon_t^{(5)}$. Let u be the machine precision, and assume that the error produced at each step is bounded by u . We have

$$|\varepsilon_t^{(1)}| \leq u(n+1) \log n.$$

Since $|\varepsilon_t^{(2)}| \leq u$ we have $|\varepsilon_t^{(3)}| \leq 2u + u(n+1) \log n$. The error produced at the 4-th step can be bounded as follows

$$|\varepsilon_t^{(4)}| \leq u + \frac{|\lambda_M^{(k)}|}{|\lambda_M^{(k)} - 1|} \frac{\varepsilon_M}{2},$$

because $\varepsilon_M/2$ bounds the probabilistic error of $\lambda_B^{(k)}$. Since

$$\varepsilon_t = \varepsilon_t^{(5)} \doteq \varepsilon + \varepsilon_t^{(3)} + \varepsilon_t^{(4)},$$

the following inequality holds

$$|\varepsilon_t| \leq u(4 + (n+1) \log n) + \frac{|\lambda_M^{(k)}| |\varepsilon_M|}{2|\lambda_M^{(k)} - 1|}.$$

In order to satisfy (4.12), we require $|\varepsilon_t| \leq \frac{1}{2}\varepsilon_A$. It is straightforward to verify that it is sufficient to take

$$u \leq \frac{1}{4(4 + n \log n)} \varepsilon_A, \quad \varepsilon_M \leq \frac{1}{8(1 + 4k\mu(A))} \varepsilon_A, \quad (4.14)$$

where $\mu(A) = \|A\|_\infty \|A^{-1}\|_\infty$, is the condition number of the matrix A .

We analyze now the probabilistic error in the computation of an approximation to $\lambda_M^{(k)}$. Let $\lambda_M^{(k)} = u_m^{(k)}/u_m^{(k-1)}$ and let $\tilde{\lambda}_M^{(k)}$ be its approximation computed during the first stage of the algorithm RANPOWER. We fix a positive threshold ε_M , we wish to determine a lower bound on the number N of Markov chains to simulate, so that $\Pr\left(|\lambda_M^{(k)} - \tilde{\lambda}_M^{(k)}| < \frac{1}{2}\varepsilon_M |\lambda_B^{(k)}|\right)$ is close to 1.

Let $S^{(i)} = \sum_{j=1}^N \Phi_m^{(i)}(\omega_j)/N$. From Chebyshev inequality, we have

$$\Pr\left(|u_m^{(k)} - S^{(k)}| > \varepsilon_1 |u_m^{(k)}|\right) \leq \frac{\text{Var}(\Phi_m^{(k)})}{N(\varepsilon_1 |u_m^{(k)}|)}, \quad (4.15)$$

where $\varepsilon_1 = (1/4)\varepsilon_M$.

The following theorem gives a bound on the m -th entry of the vector $\mathbf{u}^{(k)} = M^k \mathbf{b}$ under certain assumptions on the starting vector \mathbf{b} .

Theorem 4.2.1 *Let \mathbf{b} be a vector with positive entries, i.e. $b_i > 0$, and such that*

$$\frac{\min_{i=1,\dots,n} b_i}{\max_{i=1,\dots,n} b_i} > \left(1 + \frac{1}{4k}\right)^k - 1, \quad (4.16)$$

and let $u_m^{(k)}$ be the m -th entry of the vector $\mathbf{u}^{(k)} = M^k \mathbf{b}$. Then, for $s = 0, 1, \dots, k$, and for every index m , we have:

$$\min_{i=1,\dots,n} b_i - \max_{i=1,\dots,n} b_i \left[\left(1 + \frac{1}{4k}\right)^s - 1 \right] \leq u_m^{(s)} \leq \max_{i=1,\dots,n} b_i \left(1 + \frac{1}{4k}\right)^s.$$

Proof. We proceed by induction on s .

For $s = 0$ we have

$$u_i^{(0)} = b_i$$

and then $\min_{i=1,\dots,n} b_i \leq u_m^{(0)} \leq \max_{i=1,\dots,n} b_i$. Suppose that the theorem is true for $s-1$, i.e.

$$\min_{i=1,\dots,n} b_i - \max_{i=1,\dots,n} b_i \left[\left(1 + \frac{1}{4k}\right)^{s-1} - 1 \right] \leq u_m^{(s-1)} \leq \max_{i=1,\dots,n} b_i \left(1 + \frac{1}{4k}\right)^{s-1}.$$

By inductive hypothesis and by using equation (4.16) we claim that $u_m^{(s-1)} > 0$. By definition

$$u_m^{(s)} = u_m^{(s-1)} + \frac{1}{4k\|A\|_\infty} \sum_{j=1}^n a_{mj} u_j^{(s-1)}, \quad (4.17)$$

then equation (4.17) can be rewritten as

$$\begin{aligned} u_m^{(s)} &= u_m^{(s-1)} + \frac{1}{4k\|A\|_\infty} \left[\sum_{j:a_{mj}>0} a_{mj} u_j^{(s-1)} - \sum_{j:a_{mj}<0} |a_{mj}| u_m^{(s-1)} \right] \\ &\leq \max_i u_i^{(s-1)} + \frac{1}{4k\|A\|_\infty} \left[\max_i u_i^{(s-1)} \sum_{j:a_{mj}>0} a_{mj} - \min_i u_i^{(s-1)} \sum_{j:a_{mj}<0} |a_{mj}| \right] \\ &\leq \max_i u_i^{(s-1)} + \frac{\min_i u_i^{(s-1)}}{4k\|A\|_\infty} \left[\sum_{j=1}^n a_{mj} + \frac{\max_i u_i^{(s-1)} - \min_i u_i^{(s-1)}}{\min_i u_i^{(s-1)}} \sum_{j:a_{mj}>0} a_{mj} \right]. \end{aligned}$$

Since the sum of the elements of each row is less than the ∞ -norm, we have

$$\begin{aligned} u_m^{(s-1)} &\leq \max_i u_i^{(s-1)} + \frac{\min_i u_i^{(s-1)}}{4k\|A\|_\infty} \left[\sum_{j=1}^n a_{mj} + \frac{\max_i u_i^{(s-1)} - \min_i u_i^{(s-1)}}{\min_i u_i^{(s-1)}} \sum_{j:a_{mj}>0} a_{mj} \right] \\ &\leq \max_i u_i^{(s-1)} + \frac{\min_i u_i^{(s-1)}}{4k} + \frac{\max_i u_i^{(s-1)}}{4k} - \frac{\min_i u_i^{(s-1)}}{4k} \\ &\leq \max_i u_i^{(s-1)} \left(1 + \frac{1}{4k}\right). \end{aligned} \quad (4.18)$$

By induction hypothesis together with (4.18) we get

$$u_m^{(s)} \leq \max_{i=1,\dots,n} b_i \left(1 + \frac{1}{4k}\right)^s. \quad (4.19)$$

Let us now prove the other inequality. From equation (4.17), separating negative terms from positive, we get

$$\begin{aligned} u_m^{(s)} &= u_m^{(s-1)} + \frac{1}{4k\|A\|_\infty} \left[\sum_{j:a_{mj}>0} a_{mj} u_j^{(s-1)} - \sum_{j:a_{mj}<0} |a_{mj}| u_j^{(s-1)} \right] \\ &\geq \min_i u_i^{(s-1)} + \frac{1}{4k\|A\|_\infty} \left[\min_i u_i^{(s-1)} \sum_{j:a_{mj}>0} a_{mj} - \max_i u_i^{(s-1)} \sum_{j:a_{mj}<0} |a_{mj}| \right] \\ &= \min_i u_m^{(s-1)} + \frac{\min_i u_i^{(s-1)}}{4k\|A\|_\infty} \left[\sum_{j=1}^n a_{mj} - \frac{\max_i u_i^{(s-1)} - \min_i u_i^{(s-1)}}{\min_i u_i^{(s-1)}} \sum_{j:a_{mj}<0} |a_{mj}| \right] \end{aligned}$$

Observing that $\|A\|_\infty \geq -\sum_{j=1}^n a_{mj}$ and by inductive hypothesis, we get

$$u_m^{(s)} \geq \min_i b_i - \max_i b_i \left[\left(1 + \frac{1}{4k}\right)^s - 1 \right].$$

□

The same theorem holds when we consider \bar{M} instead of M . The proof is essentially the same of that of Theorem 4.2.1 where is used the fact that $-\sum_{j=1}^n a_{mj} \leq \|A\|_\infty$.

In order to evaluate inequality (4.15), we need to bound $\text{Var}(\Phi_m^{(k)})$. We remind that

$$\text{Var}(\Phi_m^{(k)}(\omega_i)) = E[(\Phi_m^{(k)}(\omega_i))^2] - [E(\Phi_m^{(k)}(\omega_i))]^2.$$

The following lemma allows one to bound the variance of the random variable $\Phi_m^{(k)}$. The same lemma holds for the matrix \bar{M} .

Lemma 4.2.2 *Let $M \equiv (m_{ij})$ be a matrix defined as in (4.8). Then we have*

$$1 - \frac{1}{4k} \leq \sum_{j=1}^n m_{ij} \leq 1 + \frac{1}{4k}.$$

□

The proof of the Lemma follows easily from the definition of matrix M (see equation (4.8)). By lemma 4.2.2 we have

$$\begin{aligned} |\Phi_m^{(k)}(\omega_i)| &= |f_{mi_1} f_{i_1 i_2} \cdots f_{i_{k-1} i_k} b_{i_k}| \\ &\leq \max_i b_i |\max_{ij} f_{ij}|^k \\ &= \max_i b_i \left(\max_i \sum_{h=1}^n |b_{ih}| \right)^k \\ &\leq \max_i b_i \left(1 + \frac{1}{4k} \right)^k, \end{aligned}$$

and by theorem 4.2.1 we can bound the variance as follows

$$\begin{aligned} \text{Var}(\Phi_m^{(k)}) &\leq (\max_i b_i)^2 \left(1 + \frac{1}{4k} \right)^{2k} - \left[\min_i b_i - \max_i b_i \left[\left(1 + \frac{1}{4k} \right)^k - 1 \right] \right]^2 \\ &= -(\min_i b_i + \max_i b_i)^2 + 2 \max_i b_i \left(1 + \frac{1}{4k} \right)^k (\min_i b_i + \max_i b_i) \\ &\leq 2 \max_i b_i \left(1 + \frac{1}{4k} \right)^k (\min_i b_i + \max_i b_i). \end{aligned}$$

Inequality (4.15) can be rewritten as

$$\Pr(|u_m^{(k)} - S^{(k)}| > \varepsilon_1 |u_m^{(k)}|) \leq \frac{2 \max_i b_i \left(1 + \frac{1}{4k} \right)^k (\min_i b_i + \max_i b_i)}{N(\varepsilon_1 |u_m^{(k)}|)^2}. \quad (4.20)$$

Observe that

$$\Pr\left(\frac{|\lambda_M^{(k)} - \tilde{\lambda}_M^{(k)}|}{|\lambda_M^{(k)}|} \leq \frac{1}{2} \varepsilon_M\right) = \Pr\left(\frac{|u_m^{(k)} - S^{(k)}|}{|u_m^{(k)}|} \leq \varepsilon_1, \frac{|u_m^{(k-1)} - S^{(k-1)}|}{|u_m^{(k-1)}|} \leq \varepsilon_2\right),$$

where $\varepsilon_1 = \varepsilon_2 = (1/4)\varepsilon_M$.

We notice that the sequences ω_i and ν_i are chosen independently in RANPOWER, then the random variables $S^{(k)}$ and $S^{(k-1)}$ are independent. From previous equation, we have,

$$\begin{aligned}
& \Pr\left(\frac{|\lambda_M^{(k)} - \tilde{\lambda}_M^{(k)}|}{|\lambda_M^{(k)}|} \leq \frac{1}{2}\varepsilon_M\right) = \\
& = \Pr\left(\frac{|u_m^{(k)} - S^{(k)}|}{|u_m^{(k)}|} \leq \varepsilon_1\right) \Pr\left(\frac{|u_m^{(k-1)} - S^{(k-1)}|}{|u_m^{(k-1)}|} \leq \varepsilon_2\right) \\
& > \left(1 - \frac{\text{Var}(\Phi_m^{(k)})}{N(\varepsilon_1|u_m^{(k)}|)^2}\right) \left(1 - \frac{\text{Var}(\Phi_m^{(k-1)})}{N(\varepsilon_2|u_m^{(k-1)}|)^2}\right). \tag{4.21}
\end{aligned}$$

It is straightforward to see that

$$\Pr\left(\frac{|u_m^{(k)} - S^{(k)}|}{|u_m^{(k)}|} \leq \varepsilon_1, \frac{|u_m^{(k-1)} - S^{(k-1)}|}{|u_m^{(k-1)}|} \leq \varepsilon_2\right) > 1 - \frac{\text{Var}(\Phi_m^{(k)})}{N(\varepsilon_1|u_m^{(k)}|)^2} - \frac{\text{Var}(\Phi_m^{(k-1)})}{N(\varepsilon_2|u_m^{(k-1)}|)^2}.$$

Using the bounds proved in Theorem 4.2.1, and inequalities (4.20) and (4.21), we have that, in order to have a probability of success greater than $1 - \delta$, it is sufficient to choose

$$N = \Omega(\delta^{-1}\varepsilon_A^{-2}\mu(A)k^2). \tag{4.22}$$

If A is a normal matrix, i.e. $A^T A = A A^T$, we can choose

$$N = \Omega(\delta^{-1}\varepsilon_A^{-2}nk^2). \tag{4.23}$$

In this case, $|\lambda_1^{(A)}| \geq \|A\|_\infty/\sqrt{n}$, and hence $|\lambda_1^{(M)} - 1| \geq 1/(4k\sqrt{n})$. It follows that it is sufficient to choose $\varepsilon_M \leq \frac{1}{4(1+4k\sqrt{n})}\varepsilon_A$. This bound on ε_M guarantees equality (4.23).

By (4.23) we have that, if the problem is ill-posed, e.g. $\mu(A)$ increases exponentially with the order of the matrix, it is necessary to simulate an exponential number of Markov chains. On the other hand, if A is normal, then N is linear in n .

4.3 Analysis of the Computational Cost

In this section we study the computational cost of the algorithm RANPOWER on a CREW-PRAM.

Let $T(n)$ and $P(n)$ be the parallel time and the number of processors required for an input of size n .

We first analyze the parallel cost of the step **1.** of the algorithm. This step consists of the double application of a procedure that allows to approximate $\lambda_M^{(k)}$ and $\lambda_{\tilde{M}}^{(k)}$.

The computation of f_{ij} , in stage 1.1, can be performed in $\lceil \log n \rceil$ time on $n\lceil n/\log n \rceil$ processors. Then the values p_{ij} can be computed in constant time. Similarly, stage 1.2 requires $\lceil \log n \rceil$ time and $n\lceil n/\log n \rceil$ processors. During stage 1.3 we compute in parallel $2N$ trajectories. Stage 1.3.1 can be carried out in constant time on $2nk - n$ processors. Step 1.3.2 can be performed by using a binary search algorithm with running time $\lceil \log n \rceil$ on $2nk - n$ processors. Hence, stage 1.3 requires $O(\log n)$ time on $2N(2nk - n)$ processors. By using $2Nk$ processors the values in stage 1.4 can be computed in

$\lceil \log k \rceil$ time. Stage 1.5 consists of the sums of N terms, so that $\lceil \log N \rceil$ time steps suffice on $\lceil N/\log N \rceil$ processors. Stage 1.6 can be carried out in constant time. Since stages 1.1–1.6 are performed twice, the global cost of stage 1 is

$$\begin{aligned} T(n) &\leq 6\lceil \log n \rceil + 2\lceil \log k \rceil + 2\lceil \log N \rceil + O(1), \\ P(n) &\leq \max\{n\lceil \log n \rceil, 2N(2nk - n)\}. \end{aligned}$$

Step 2 requires the computation of $\|A\|_\infty$, that can be performed in time $\lceil \log n \rceil$ on $n\lceil n/\log n \rceil$ processors. Finally, the value $\tilde{\lambda}_1^{(A)}$ in step 3 can be computed in constant time. The cost of the algorithm RANPOWER is

$$\begin{aligned} T(n) &\leq 7\lceil \log n \rceil + 2\lceil \log k \rceil + 2\lceil \log N \rceil + O(1), \\ P(n) &\leq \max\{n\lceil \log n \rceil, 2N(2nk - n)\}. \end{aligned}$$

Theorem 4.3.1 *Let A be a symmetric $n \times n$ matrix with real entries. Assume we want to compute an approximation of the dominant eigenvalue of A with a total error at most of ε_A , by using algorithm RANPOWER. If A is normal or its condition number $\mu(A)$ increases at most polynomially with the order of the matrix, then RANPOWER computes an ε_A -approximation of the dominant eigenvalue in parallel time $T(n) = O(\log n + \log k)$ and $P(n) = kn^{O(1)}$, with high probability.*

Proof. We observe that the bound on the parallel cost follows from the hypotheses of the theorem. In fact in this case, by equations (4.22) and (4.23), we have $N = O(n^h)$, and then $\log N = O(\log n)$. \square

This theorem proves that the parallel time cost of the randomized algorithm RANPOWER is lower than that of the deterministic implementation of the power method.

Chapter 5

Conclusions

This thesis has investigated the effectiveness of randomized techniques for computing the largest eigenvalue and a corresponding eigenvector of a symmetric matrix. In particular, our study has been addressed towards two different directions. We first asked if the use of randomized techniques allows one to characterize better the behavior of very well known methods. Then, we considered the problem of designing a randomized algorithm whose performance is better than that of the best deterministic implementation of the power method.

Both of these questions have been answered positively. In particular, in Chapter 2 and 3 we considered the power and Lanczos methods when the choice of the starting vector is done at random. We noted that this is usually the case, in practical implementation of these methods. We defined the randomized error in the sense of \mathcal{L}_p , for $p \in [1, +\infty]$, and we analyzed the behavior of these methods for every value of the norm parameter p . Our analysis has pointed out mainly two properties of Krylov methods that were not known in the deterministic setting.

- In the randomized setting, the behavior of the above mentioned methods as well as of any other polynomial methods differs when eigenvector or eigenvalue estimate is considered. In fact, for eigenvector estimate it is impossible to give sharp bounds that do not depend on the matrix ¹, while for eigenvalue these bounds exist ². This allows one to conclude that eigenvector estimate is a harder problem than eigenvalue estimate in the setting considered in this thesis.
- From the analysis of distribution dependent bounds ³ it is evident that the speed of convergence of power and Lanczos methods depends on the relation between the multiplicity r of the largest eigenvalue and the norm parameter p . In particular, the error decreases faster when the dimension r of the eigenspace corresponding to λ_1 is large, but it slows down when the parameter p increases. This is a complete new fact respect to the asymptotic deterministic case, where the speed of convergence depends only on the eigenvalues of the matrix and on the iteration step.

¹See Sections 2.3.1 and 3.1.

²See Sections 2.2.1, 3.4.1.

³See Sections 2.2.2, 2.4, 3.3 and 3.4.2.

In Chapter 4 we described a parallel randomized algorithm for the approximation to the largest eigenvalue. That method is based on the power algorithm, and it simulates k steps of that algorithm at each time. The cost analysis showed that, the parallel cost of our method on a CREW-PRAM with a polynomial number of processors, is $O(\log n + \log k)$ where the best deterministic implementation of the power algorithm requires in the same model $O(\log n \log k)$.

This thesis showed very encouraging results, and highlights several topics that deserve further investigations. For example, we plan to study the randomized error of Lanczos algorithm for the approximation of the first s largest eigenvalues of a symmetric positive definite matrix, $s \ll n$. For this problem error estimates are known in the deterministic case, but so far no results are available in the randomized setting.

It would be interesting also to extend our results by considering different error criteria, for example the randomized residual error defined in [LW96], or to study other methods such as Arnoldi method in this setting.

Bibliography

- [BCM88] D. Bini, M. Capovani, and O. Menchi. *Metodi Numerici per l'Algebra Lineare*. Zanichelli, Bologna, 1988.
- [BM58] G. E. P. Box and M. E. Muller. A Note on the Generation of Random Normal Deviates. *Ann. Math. Statist.*, 29:610–611, 1958.
- [Bro56] G. W. Brown. *Modern Mathematics for the Engineer*. McGraw-Hill, New York, 1956. Editor E. F. Beckenbach.
- [CF89] B. Codenotti and F. Flandoli. A Monte Carlo method for the parallel solution of linear systems. *J. of Complexity*, 5(1):107–117, 1989.
- [Cho87] A. W. Chow. On the Optimality of Krylov Information. *J. of Complexity*, 3:26–40, 1987.
- [CL75] F. Chatelin and J. Lemordant. La méthode de Rayleigh-Ritz appliquée à des opérateurs différentiels elliptiques. *Numer. Math*, 23:215–222, 1975.
- [CL91] B. Codenotti and M. Leoncini. Matrix Inversion in RNC^1 . *Journal of Complexity*, 7(3):282–295, 1991.
- [dCM96] G. M. Del Corso and G. Manzini. On the Randomized Error of Polynomial Methods for Eigenvector and Eigenvalue Estimate. Technical Report B4-96-15, Istituto di Matematica Computazionale, CNR, Pisa, 1996.
- [dCo95] G. M. Del Corso. Randomization and the Parallel Solution of Linear Algebra Problems. *Comp. Math. Appl.*, 30(11):59–72, 1995.
- [dCo96] G. M. Del Corso. Estimating an Eigenvector by the Power Method with a Random Start. *SIAM J. on Matrix Anal. Appl.*, 1996. To appear.
- [Dix83] J. D. Dixon. Estimating extremal Eigenvalues and Condition Numbers of Matrices. *SIAM J. Numer. Anal.*, 20:812–814, 1983.
- [Fel68] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 1. John Wiley & Sons Inc., 1968.

- [FW78] S. Fortune and J. Wyllie. Parallelism in Random Access Machines. In *Proc. 10th ACM Symposium on Theory of Computing (STOC)*, pages 114–118, 1978.
- [GL89] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, 1989.
- [GR94] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, 1994.
- [Kan66] S. Kaniel. Estimates for Some Computational Techniques in Linear Algebra. *Math. Comp.*, 20:369–378, 1966.
- [Knu81] D. E. Knuth. *The Art of Computer Programming: Seminumerical Algorithms*. Addison-Wesley, 1981. Second edition.
- [Kos88] E. Kostlan. Complexity Theory of Numerical Linear Algebra. *J. Computational and Applied Mathematics*, 22:219–230, 1988.
- [Kos91] E. Kostlan. Statistical Complexity of Dominant Eigenvector Calculation. *J. of Complexity*, 7:371–379, 1991.
- [KP76] W. Kahan and B. N. Parlett. How far should we go with Lanczos process? In J. Bunch and D. Rose, editors, *Sparse Matrix Computations*, pages 131–144. Academic Press, New York, 1976.
- [Kuc86] J. Kuczyński. On the Optimal Solution of large Eigenpair Problems. *J. Complexity*, 2:131–162, 1986.
- [KW92a] J. Kuczyński and H. Woźniakowski. Estimating the Largest Eigenvalue by the Power and Lanczos Algorithms with a Random Start. *SIAM J. Matrix Anal. Appl.*, 13:1094–1122, 1992.
- [KW92b] J. Kuczyński and H. Woźniakowski. Power method with a random start for approximating an eigenvector belonging to the largest eigenvalue. Unpublished, 1992.
- [KW94] J. Kuczyński and H. Woźniakowski. Probabilistic Bounds on the Extremal Eigenvalues and Condition Number by the Lanczos Algorithm. *SIAM J. Matrix Anal. Appl.*, 15:672–691, 1994.
- [Lan69] S. Lang. *Analysis II*. Addison-Wesley Publishing Company, 1969.
- [LS94] Z. Leyk and D. E. Stewart. Meschach: Matrix computation in c. In *Proc. of the CMA*, volume 32. Australian National University, 1994.
- [LSV79] D. P. Leary, G. W. Stewart, and J. S. Vandergraft. Estimating the Largest Eigenvalue of a Positive Definite Matrix. *Math. of Computation*, 33:1289–1292, 1979.

- [LW96] Z. Leyk and H. Woźniakowski. Estimating a Largest Eigenvector by Polynomial Algorithms with a Random Start. Technical Report CUCS-026-96, Columbia University, New York, 1996.
- [Nov88] E. Novak. *Deterministic and Stochastic Error Bounds in Numerical Analysis*. Springer-Verlag, 1988. Lecture Notes in Mathematics N.1349.
- [Pai71] C. C. Paige. *The Computation of Eigenvalues and Eigenvectors of very Large Sparse Matrices*. PhD thesis, University of London, 1971.
- [Pai72] C. C. Paige. Computational Variants of the Lanczos Method for the Eigenproblem. *J. Inst. Math. Appl.*, 10:373–381, 1972.
- [Pai76] C. C. Paige. Error Analysis of the Lanczos Algorithm for Tridiagonalizing a Symmetric Matrix. *J. Inst. Math. Appl.*, 18:341–349, 1976.
- [Par80] B. N. Parlett. *The Symmetric Eigenvalue Problem*. Prentice-Hall, Englewood Cliffs, 1980.
- [PSS82] B. N. Parlett, H. Simon, and L. M. Stringer. On estimating the Largest Eigenvalue with the Lanczos Algorithm. *Math. of Computation*, 38:153–165, 1982.
- [Rud66] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, 1966.
- [Saa80] Y. Saad. On the Rates of Convergence of the Lanczos and the Block Lanczos Methods. *SIAM J. Numer. Anal.*, 17:687–706, 1980.
- [Saa92] Y. Saad. *Numerical Methods for Large Eigenvalues Problems*. Manchester University Press, Manchester, 1992.
- [Sco78] D. S. Scott. *Analysis of the Symmetric Lanczos Process*. PhD thesis, University of California at Berkeley, 1978.
- [Shu86] M. Shub. The Geometry and Topology of Dynamical Systems and Algorithms for Numerical Problems. In Liao Shantao, editor, *Proc. of the 1983 Beijing Symposium on Differential Geometry and Differential Equations*, Beijing, China, 1986. Science Press.
- [TW84] J. F. Traub and H. Woźniakowski. On the Optimal Solution of Large Linear Systems. *Journal of ACM*, 31:545–559, 1984.
- [TWW88] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity*. Academic Press, 1988.
- [Wil65] J. H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, London, 1965.
- [Wri89] P. E. Wright. Statistical Complexity of the Power Method for Markov Chains. *J. of Complexity*, 5:119–143, 1989.