

An implicit multishift QR-algorithm for Hermitian plus low-rank matrices

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1 The Problem

2 An $\mathcal{O}(n)$ representation

3 The algorithm

4 Deflation

5 Numerical Experiments

6 Conclusion



The Problem

- Eigenvalues computation of structured rank matrices
- QR on structures
- Keyword is **representation preserved by QR -steps**



The Problem

Efficient computation of the eigenvalues of a **Hermitian plus a low rank correction**

- Perfectly Hermitian structure perturbed by a low rank correction
- Comrade matrices
- Hamiltonian-like matrices



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Givens-weight representation

We will represent matrices by means their ***QR* factorization**.

A 6×6 Hessenberg matrix H

1	X	X	X	X	X	X
2		X	X	X	X	X
3			X	X	X	X
4				X	X	X
5					X	X
6						X

$$H = G_1 G_2 \dots G_5 R.$$



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1		x	x	x	x	x	x
2	0	x	x	x	x	x	x
3		x	x	x	x	x	x
4			x	x	x	x	x
5				x	x	x	x
6					x	x	x
	1						

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$$\begin{array}{c|cccccc} & & \times & \times & \times & \times & \times \\ \textcircled{1} & \nearrow & 0 & \times & \times & \times & \times \\ \textcircled{2} & \nearrow & 0 & \times & \times & \times & \times \\ \textcircled{3} & \nearrow & 0 & \times & \times & \times & \times \\ \textcircled{4} & \nearrow & & 0 & \times & \times & \times \\ \textcircled{5} & & & & \times & \times & \times \\ \textcircled{6} & & & & & \times & \times \end{array}$$

3 2 1

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Reduction to Hessenberg

$A = S + UV^H$, where S is Hermitian and $U, V \in C^{n \times m}$

- Reduce A into Hessenberg form.
If S has some structure the cost is $\mathcal{O}(n^2)$
- $H = Q^H A Q = Q^H S Q + Q^H U V^H Q = \hat{S} + \hat{U} \hat{V}^H$
- Let $\mathcal{F}_{n,m}$ the class of Hessenberg matrices which are the sum of a $n \times n$ Hermitian plus rank- m matrix



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QR-algorithm

- $\mathcal{F}_{n,m}$ is closed under QR steps
- Multishift QR

$$\begin{aligned} p_d^{(k)}(H^{(k)}) &= Q^{(k)} R^{(k)} \\ H^{(k+1)} &= Q^{(k)H} H^{(k)} Q^{(k)}, \end{aligned}$$

- For $d = 2$, $p_2^{(k)}(H^{(k)}) = (H^{(k)} - \mu_1^{(k)} I)(H^{(k)} - \mu_2^{(k)} I)$



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$$H^{(k+1)} = Q^{(k)H} H^{(k)} Q^{(k)},$$

where, for every k , $p_d^{(k)}(x)$ is a monic polynomial of degree d .

- For $d = 2$, $p_2^{(k)}(H^{(k)}) = (H^{(k)} - \mu_1^{(k)} I)(H^{(k)} - \mu_2^{(k)} I)$



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The sequence $\{H^{(k)}\}_k$ converges to the real canonical Schur form of H .

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An implicit method

Perform the transition $H^{(k)} \rightarrow H^{(k+1)}$ implicitly.

- **Initialization step** Compute $Q_{\mathcal{I}}$ such that

$$Q_{\mathcal{I}} p_d(H) \mathbf{e}_1 = \beta \mathbf{e}_1, \quad \beta = \|p_d(H) \mathbf{e}_1\|_2,$$

$H_{\mathcal{I}} = Q_{\mathcal{I}}^H H Q_{\mathcal{I}}$ non Hessenberg. Has a bulge with tip in position $(d + 2, 1)$.

- **Chasing steps** Chase the bulge down to restore the Hessenberg structure. Let Q_C the orthogonal matrix obtained, product of $d(n - 2)$ Givens factors.



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An $\mathcal{O}(n)$ representation

$H = S + UV^H$, hence

$$S = \begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot \\ \blacksquare & \times & \times & \cdot & \cdot & \cdot \\ \blacksquare & \blacksquare & \times & \times & \cdot & \cdot \\ \blacksquare & \blacksquare & \blacksquare & \times & \times & \cdot \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \times & \times \end{bmatrix}$$



Rank-1 perturbation

If $H = S + \mathbf{u}\mathbf{v}^H$, we can annihilate most of the rank-1 part of S as follows

$$\begin{array}{c|cccccc} \textcircled{1} & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \textcircled{2} & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcircled{3} & \boxtimes & \times & \times & \cdot & \cdot & \cdot \\ \textcircled{4} & \boxtimes & \boxtimes & \times & \times & \cdot & \cdot \\ \textcircled{5} & \curvearrowright & \otimes & \otimes & \boxtimes & \times & \times \\ \textcircled{6} & & & & & \times & \times & \times \\ \hline & & & & & 1 & & \end{array}$$



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Rank-1 perturbation

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$$\begin{array}{c|cccccc} & & & & & & \\ \textcircled{1} & & \times & \cdot & \cdot & \cdot & \cdot \\ \textcircled{2} & & \times & \times & \cdot & \cdot & \cdot \\ \textcircled{3} & & \times & \times & \times & \cdot & \cdot \\ \textcircled{4} & & & \times & \times & \times & \cdot \\ \textcircled{5} & & & & \times & \times & \times \\ \textcircled{6} & & & & & \times & \times & \times \\ \hline & 3 & 2 & 1 & & & & \end{array}$$

Diagram illustrating the rank-1 update of a matrix. A vertical column of circled numbers 1 through 6 is on the left, corresponding to rows in the matrix. Brackets on the left indicate the columns being updated: a single bracket for row 1, a double bracket for rows 2 and 3, a triple bracket for rows 4, 5, and 6. The matrix itself has a diagonal of 'x' characters and zeros elsewhere, except for the rank-1 update which has 'x' characters in the (i,i) position for each i from 1 to 6.



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$$\begin{array}{c|cccccc}
 & \textcircled{1} & & \times & \cdot & \cdot & \cdot & \cdot \\
 & \textcircled{2} & & \times & \times & \cdot & \cdot & \cdot \\
 & \textcircled{3} & & \times & \times & \times & \cdot & \cdot \\
 & \textcircled{4} & & & \times & \times & \times & \cdot \\
 & \textcircled{5} & & & & \times & \times & \times \\
 & \textcircled{6} & & & & & \times & \times & \times \\
 \hline & & 3 & 2 & 1 & & & &
 \end{array}$$

We represent S as $S = \tilde{V}\tilde{B}$, where $\tilde{V} = G_1 G_2 G_3$, \tilde{B} is the generalized Hessenberg.



Representation of H

$$\begin{aligned}
 H &= S + \mathbf{u}\mathbf{v}^H = \tilde{V} \left(\tilde{B} + \tilde{\mathbf{u}}\mathbf{v}^H \right) \\
 &= \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \end{bmatrix} \mathbf{v}^H \right).
 \end{aligned}$$



Representation of H

To simplify the initialization steps of the implicit QR

$$H = V \left(B + \hat{\mathbf{u}}\mathbf{v}^H \right)$$

$$= \begin{pmatrix} & \end{pmatrix} \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot \\ & \times & \times & \times & \cdot & \cdot \\ & & \times & \times & \times & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ 0 \\ 0 \end{bmatrix} \mathbf{v}^H \right)$$



Rank-2 perturbation

If the perturbation is of rank 2,

$$S = H - \mathbf{u}\mathbf{v}^H - \mathbf{x}\mathbf{y}^H$$

We need two sequences of ascending Givens transformations to peel off the rank-2 part in S .



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1 2 3 4 5 6 7	X X X X X X X X X X X X X X X . . X X X X X . X X X X X
	4 3 2 1



Representation of H

$$H = \left(\begin{array}{c} \left[\begin{array}{cccccc} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ & x & x & x & x & \cdot & \cdot \\ & & x & x & x & x & \cdot \\ & & & x & x & x & x \end{array} \right] + \left[\begin{array}{c} x \\ \otimes_2 \\ \otimes_1 \end{array} \right] v^H + \left[\begin{array}{c} x \\ x \\ \otimes_4 \\ \otimes_3 \end{array} \right] y^H \end{array} \right)$$



Representation of H

The elements \otimes in the vectors still need to be removed!

$$H = \left(\begin{array}{c} \left[\begin{array}{cccccc} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot \\ \times & \times & \times & \times & \times & \times & \times \end{array} \right] + \left[\begin{array}{c} \times \\ \otimes_2 \\ \otimes_1 \end{array} \right] \mathbf{v}^H + \left[\begin{array}{c} \times \\ \times \\ \otimes_4 \\ \otimes_3 \end{array} \right] \mathbf{y}^H \end{array} \right)$$



Representation of H

$$H = \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{v}^H + \begin{bmatrix} \times \\ \times \\ 0 \\ 0 \end{bmatrix} \mathbf{y}^H \right)$$



Cost of the representation

In the general rank- m perturbation

$$H = \prod_{i=1}^m V^{(\mathbf{u}_i)} \left(B + \sum_{i=1}^m \mathbf{u}_i \mathbf{v}_i^H \right),$$

- B is a generalized Hessenberg with $m+1$ subdiagonals
- \mathbf{u}_i are sparse vectors with only the first i entries nonzero
- each $V^{(\mathbf{u}_i)}$ consists of $n-i$ Givens transformations,
- $\sum_{i=1}^m (n-i) = \mathcal{O}(nm + m^2)$ Givens transformations;
- $\sum_{i=1}^{m+2} (n+1-i) = \mathcal{O}(nm + m^2)$ entries for the matrix B ;
- $\sum_{i=1}^m i = \mathcal{O}(m^2)$ entries for the vectors \mathbf{u}_i .



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The algorithm

- We present the **single shift** QR applied to $H \in \mathcal{F}_{n,2}$
- The generalization to multishift and/or $m > 2$ has too many technical details ...

... the details can be found in the paper!



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Initialization

- Find $Q_{\mathcal{I}}$ such that

$$Q_{\mathcal{I}}^H(H - \mu I)\mathbf{e}_1 = \beta \mathbf{e}_1$$

A single Givens is enough for initialization!

$Q_{\mathcal{I}} = G_1$, acting on rows 1 and 2



Initialization

- Find G_1 such that

$$G_1^H(H - \mu I)\mathbf{e}_1 = \beta \mathbf{e}_1$$

A single Givens is enough for initialization!

$Q_{\mathcal{I}} = G_1$, acting on rows 1 and 2



Initialization

$$H_2 = G_1^H H_1 G_1$$



Initialization

$$H_2 = G_1^H V_1^{(\mathbf{u})} V_1^{(\mathbf{x})} (B_1 + \mathbf{u}_1 \mathbf{v}_1^H + \mathbf{x}_1 \mathbf{y}_1^H) G_1$$



Inizialization

$$H_2 = G_1^H V_1^{(\mathbf{u})} V_1^{(\mathbf{x})} (B_1 + \mathbf{u}_1 \mathbf{v}_1^H + \mathbf{x}_1 \mathbf{y}_1^H) G_1$$

$$H_2 = \left(\begin{array}{c} \text{Diagram showing a 2x2 block structure with arrows indicating row and column indices. A red 'X' is placed above the first row of the matrix.} \\ \left[\begin{array}{cccccc} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot \end{array} \right] + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_1^H \end{array} \right) G_1$$



Inizialization

$$H_2 = G_1^H V_1^{(\mathbf{u})} V_1^{(\mathbf{x})} (B_1 + \mathbf{u}_1 \mathbf{v}_1^H + \mathbf{x}_1 \mathbf{y}_1^H) G_1$$

$$H_2 = \left(\begin{bmatrix} \times & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_1^H \right) G_1$$



Inizialization

$$H_2 = G_1^H V_1^{(\mathbf{u})} V_1^{(\mathbf{x})} (B_1 + \mathbf{u}_1 \mathbf{v}_1^H + \mathbf{x}_1 \mathbf{y}_1^H) G_1$$

$$H_2 = \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{\times} & \textcolor{red}{\times} & \textcolor{red}{\times} & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{\times} & \textcolor{red}{\times} & \textcolor{red}{\times} & \textcolor{red}{\times} & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ & & & & \times & \cdot & \cdot \\ & & & & \times & \times & \cdot \\ & & & & \times & \times & \times \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_1^H \right) G_1$$

$$H_2 = \tilde{V}_1^{(\mathbf{u})} \tilde{V}_1^{(\mathbf{x})} (\tilde{B}_1 + \mathbf{u}_1 \mathbf{v}_1^H + \mathbf{x}_1 \mathbf{y}_1^H) G_1$$

Vectors \mathbf{u}_1 and \mathbf{x}_1 don't change during the iterations



Transformation on the right

Apply transformation G_1 on the right

$$H_2 = \left(\begin{array}{c} \left[\begin{array}{cccccc} x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & \cdot \\ x & x & x & x & x & x \end{array} \right] + \left[\begin{array}{c} x \\ \vdots \\ \vdots \\ v_2^H \\ \vdots \\ \vdots \end{array} \right] + \left[\begin{array}{c} x \\ \vdots \\ \vdots \\ y_2^H \\ \vdots \\ \vdots \end{array} \right] \end{array} \right).$$



Transformation on the right

Apply transformation G_1 on the right

$$H_2 = \left(\begin{array}{c} \left[\begin{array}{cccccc} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & \cdot \\ x & x & x & x & x & x & x \end{array} \right] + \left[\begin{array}{c} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right] v_2^H + \left[\begin{array}{c} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right] y_2^H \end{array} \right).$$



Transformation on the right

Apply transformation G_1 on the right

$$H_2 = \left(\begin{array}{c} \text{Diagram showing a 7x7 matrix } H_2 \text{ with a circled 'X' at position (5,1). Above the matrix is a sequence of 7 right-pointing arrows, each pointing to the next column from left to right.} \\ \left[\begin{array}{cccccc} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ \textcircled{X} & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & \end{array} \right] + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} v_2^H + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} y_2^H \end{array} \right).$$

We change vectors v_1 and y_1 and a **bulge** is created in B



Transformation on the right

To remove the undesired \otimes another Givens is needed

$$H_2 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ 0 & x & x & x & x & \cdot & \cdot \\ & x & x & x & x & \cdot & \cdot \\ & x & x & x & x & x & \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{y}_2^H \right)$$



Transformation on the right

To remove the undesired \otimes another Givens is needed

$$H_2 = \left(\begin{bmatrix} x & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot & \cdot \\ 0 & x & x & x & x & \cdot & \cdot & \cdot \\ & x & x & x & x & x & \cdot & \cdot \\ & & x & x & x & x & x & \cdot \\ & & & x & x & x & x & x \end{bmatrix} + \begin{bmatrix} x \\ x \\ x \\ v_2^H \\ \vdots \\ y_2^H \end{bmatrix} \right)$$



Transformation on the right

To remove the undesired \otimes another Givens is needed

$$H_2 = \begin{matrix} \cancel{\otimes} \\ \text{Givens } \end{matrix} \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & \cdot \\ x & x & x & x & x & x & x \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{y}_2^H \right)$$



Transformation on the right

To remove the undesired \otimes another Givens is needed

$$H_2 = \begin{array}{c} \text{⊗} \\ \text{⊗} \\ \text{⊗} \\ \text{⊗} \\ \text{⊗} \\ \text{⊗} \\ \text{⊗} \end{array} \left(\begin{bmatrix} x & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & x & \cdot \\ x & x & x & x & x & x & x & x \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} \mathbf{y}_2^H \right)$$

The Givens transformation G_2 on the left will determine the next similarity transformation.

$$H_2 = G_2 V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H)$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

$$H_3 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & \cdot \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{y}_2^H \right) \cancel{\mathbf{x}}$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

$$H_3 = \left(\begin{bmatrix} x & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{\otimes} & x & x & x & x & x & \cdot & \cdot \\ x & x & x & x & x & x & x & \cdot \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ \textcolor{red}{v}_3^H \\ \vdots \\ y_3^H \end{bmatrix} \right)$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

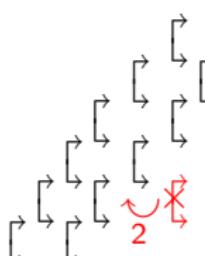
$$H_3 = \left(\begin{bmatrix} x & \cdot \\ x & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & \cdot & \cdot & \cdot & \cdot \\ x & x & x & x & x & \cdot & \cdot & \cdot \\ 0 & x & x & x & x & x & \cdot & \cdot \\ & x & x & x & x & x & x & \cdot \end{bmatrix} + \begin{bmatrix} x \\ x \\ x \\ v_3^H \\ \vdots \end{bmatrix} \right) y_3^H$$



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

$$H_3 = \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ 0 & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_3^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_3^H \right)$$


The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

$$H_3 = \cancel{\begin{array}{cccccc} \cancel{x} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot \\ 0 & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} \end{array}} \left(\begin{bmatrix} \cancel{x} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot \\ 0 & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} \end{bmatrix} + \begin{bmatrix} \cancel{x} \\ \vdots \\ \cancel{x} \end{bmatrix} \mathbf{v}_3^H + \begin{bmatrix} \cancel{x} \\ \vdots \\ \cancel{x} \end{bmatrix} \mathbf{y}_3^H \right)$$

The resulting Givens transformation G_3 acts on rows 3 and 4 and has moved down one position w.r.t. G_2 .



The Chasing

Idea: move down the Givens on the left!

$$H_3 = G_2^H H_2 G_2 = V_2^{(\mathbf{u})} V_2^{(\mathbf{x})} (B_2 + \mathbf{u}_2 \mathbf{v}_2^H + \mathbf{x}_2 \mathbf{y}_2^H) G_2$$

$$H_3 = \cancel{\begin{array}{cccccc} \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \\ \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \\ \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \\ \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \\ \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \\ \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} & \cancel{\text{X}} \end{array}} \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot \\ 0 & \times & \times & \times & \times & \cdot & \cdot \\ \times & \times & \times & \times & \times & \times & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_3^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_3^H \right)$$

$$H_3 = \textcolor{red}{G_3} V_3^{(\mathbf{u})} V_3^{(\mathbf{x})} (B_3 + \mathbf{u}_3 \mathbf{v}_3^H + \mathbf{x}_3 \mathbf{y}_3^H)$$



The last transformations

The implicit algorithm cannot run fully to the end...

The last few columns need to be computed explicitly

Using the representation we can retrieve the columns easily



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The implicit algorithm cannot run fully to the end...

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Using the representation we can retrieve the columns easily



The multishift strategy

- The multishift strategy proceeds analogously.
- $2(n - 1)$ Givens needed for the double shift case.



Deflation

- QR-iterations always combined with a deflation technique
- H is numerically of the form

$$H = \left[\begin{array}{c|c} H_1 & \times \\ \hline 0 & H_2 \end{array} \right] = \left[\begin{array}{cccc|cccc} \times & \times \\ \times & \times \\ \times & \times \\ \hline & & & & \times & \times & \times & \times \\ & & & & \textcolor{red}{0} & \times & \times & \times \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \\ & & & & & & & \times & \times \end{array} \right]$$



Deflation

- QR-iterations always combined with a deflation technique
- H is numerically of the form

$$H = \left[\begin{array}{c|c} H_1 & \times \\ \hline 0 & H_2 \end{array} \right] = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \hline & & & \\ & & & \\ & & & \\ & & & \textcolor{red}{0} \\ & & & \times & \times & \times & \times \\ & & & \times & \times & \times & \times \\ & & & \times & \times & \times & \times \\ & & & \times & \times & \times & \times \\ & & & & & \times & \times & \times \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \end{array} \right]$$

Decouple the problem and apply the QR to H_1 and H_2



Deflation

- QR-iterations always combined with a deflation technique
 - H is numerically of the form

$$H = \left[\begin{array}{c|c} H_1 & \times \\ \hline 0 & H_2 \end{array} \right] = \left[\begin{array}{cccc|cccc} \times & \times \\ \times & \times \\ & & & & \times & \times & \times & \times \\ & & & & \times & \times & \times & \times \\ \hline & & & & & & 0 & \\ & & & & & & \times & \times & \times & \times \\ & & & & & & \times & \times & \times & \times \\ & & & & & & & \times & \times & \times \\ & & & & & & & & \times & \times \\ & & & & & & & & & \times & \times \end{array} \right]$$

How to detect this using the representation?



Detecting deflation

Let $H = V^{(u)}V^{(x)}(B + \mathbf{u}\mathbf{v}^H + \mathbf{x}\mathbf{y}^H)$.

Let $W = V^{(x)^H}V^{(u)^H}H = B + \mathbf{u}\mathbf{v}^H + \mathbf{x}\mathbf{y}^H$.

$H(i+1, i) = 0$ if and only if $W(i+3, i) = 0$

$$H = \left(\begin{bmatrix} \times & \cdot & \cdot & \cdot & | & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \cdot & \cdot & | & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & | & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & | & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & | & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & | & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ | & \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ | & \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \\ \times \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} \times \\ \vdots \\ \times \\ \times \end{bmatrix} \mathbf{y}_2^H \right)$$



Performing QR -steps

$$H = \left[\begin{array}{c|cc} H(1:i-1, 1:i-1) & \times & \times \\ \hline 0 & H(i:j, i:j) & \times \\ \hline 0 & 0 & H(j+1:n, j+1:n) \end{array} \right]$$

It is possible to perform QR -steps on the different submatrices, even if they interact in the representation

Different strategies for the top, middle and bottom matrices



Performing QR -steps

$$H = \left[\begin{array}{c|c|c} H(1:i-1, 1:i-1) & \times & \times \\ \hline 0 & H(i:j, i:j) & \times \\ \hline 0 & 0 & H(j+1:n, j+1:n) \end{array} \right]$$

It is possible to perform QR -steps on the different submatrices, even if they interact in the representation

Different strategies for the **top**, middle and bottom matrices



Performing QR -steps

$$H = \left[\begin{array}{c|cc|c} H(1:i-1, 1:i-1) & & & \times \\ \hline 0 & H(i:j, i:j) & & \times \\ \hline 0 & 0 & H(j+1:n, j+1:n) & \end{array} \right]$$

It is possible to perform QR -steps on the different submatrices, even if they interact in the representation

Different strategies for the top, **middle** and bottom matrices



Performing QR -steps

$$H = \left[\begin{array}{c|cc|c} H(1:i-1, 1:i-1) & & & \times \\ \hline 0 & H(i:j, i:j) & & \times \\ \hline 0 & 0 & & H(j+1:n, j+1:n) \end{array} \right]$$

It is possible to perform QR -steps on the different submatrices, even if they interact in the representation
Different strategies for the top, middle and **bottom** matrices



A QR-step on the middle block

$$H = \left[\begin{array}{c|ccc|ccc} \times & \times \\ \times & \times \\ \hline & 0 & \times & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \times & \times & \times \\ & & & \times & \times & \times & \times & \times \\ \hline & & & & 0 & \times & \times & \times \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \end{array} \right]$$



A QR-step on the middle block

$$H = \left(\begin{bmatrix} x & \cdot \\ \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot \\ \cdot & x \end{bmatrix} + \begin{bmatrix} x \\ \cdot \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \cdot \end{bmatrix} \mathbf{y}_2^H \right)$$

The matrix H is shown in a block-diagonal form. The main diagonal block is a tridiagonal matrix with x on the main diagonal and \cdot on the super-diagonal and sub-diagonal. There are two red zeros in the matrix: one at the intersection of the 5th row and 5th column, and another at the intersection of the 8th row and 8th column. To the right of the matrix is a summand involving a vector \mathbf{v}_2^H and a vector \mathbf{y}_2^H , both represented by vertical brackets with x at the top and bottom. The entire expression is enclosed in large parentheses.

A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \begin{array}{c} \cancel{\text{X}} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \end{array} \left(\begin{bmatrix} \times & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & & & & & & & \\ \hline & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ & \times & \times & \times & \times & \cdot & \cdot & \cdot \\ & & & & \textcolor{red}{0} & \times & \times & \times \end{bmatrix} + \begin{bmatrix} \times \\ \times \\ \vdots \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} \times \\ \times \\ \vdots \end{bmatrix} \mathbf{y}_1^H \right) G_1$$

G_1 acts on rows 3 and 4



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \begin{array}{c} \text{Diagram showing a } 3 \times 3 \text{ matrix with a circled '2' at position (2,1) and a circled '0' at position (3,1). Arrows indicate row and column operations.} \\ \left(\begin{bmatrix} x & \cdot \\ \cdot & x & \cdot \\ \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot \\ \cdot & x & \cdot \\ \cdot & x \end{bmatrix} + \begin{bmatrix} x \\ \cdot \\ \cdot \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} x \\ x \\ \cdot \end{bmatrix} \mathbf{y}_1^H \right) G_1 \end{array}$$

G_1 acts on rows 3 and 4



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cancel{0} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cdot & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cdot & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cdot & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cdot & \cancel{0} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} x \\ \vdots \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} x \\ \vdots \end{bmatrix} \mathbf{y}_1^H \right) G_1$$



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} + \begin{bmatrix} x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \mathbf{v}_1^H + \begin{bmatrix} x \\ \cdot \\ \cdot \end{bmatrix} \mathbf{y}_1^H \right) G_1$$



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & x \end{bmatrix} + \begin{bmatrix} x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} v_2^H + \begin{bmatrix} x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} y_2^H \right)$$



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{bmatrix} x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cancel{0} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cdots & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot & \cdot \\ \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot & \cdot \\ \cancel{\otimes} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cdot \\ \cancel{0} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} \end{bmatrix} + \begin{bmatrix} x \\ \vdots \\ \vdots \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \vdots \\ \vdots \end{bmatrix} \mathbf{y}_2^H \right)$$



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{array}{c|c} \begin{matrix} \nearrow & \nearrow \\ \nearrow & \nearrow \end{matrix} & \begin{matrix} \times & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \hline \cdots & \cdots \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \end{matrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_2^H \end{array} \right)$$

To remove the bulge, we need a Givens acting on rows 6,7



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

- Initialization:

$$H_2 = \left(\begin{bmatrix} x & \cdot \\ \cdot & x & x & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x & x & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x & x & x & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x & x & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & x & x & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & x \\ \cdot & x \end{bmatrix} + \begin{bmatrix} x \\ \cdot \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} x \\ \cdot \\ \cdot \end{bmatrix} \mathbf{y}_2^H \right)$$

2



A QR-step on the middle block

The middle block is 3×3 . We need one Givens for the initialization step and one for the chasing!

$$H_2 = \begin{pmatrix} \text{[Diagram with red X]} & \left(\begin{bmatrix} \times & \cdot \\ \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \times & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \\ \textcolor{red}{0} & \times & \times & \times & \cdot & \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{v}_2^H + \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix} \mathbf{y}_2^H \right) \end{pmatrix}$$

The Givens matrix on the left is the one determinig the chasing step!!



Computational cost

A single QR -step

- Inizialization: ≈ 69 flops
- Bulge chasing: $\approx 134(n - 2)$ flops



Error criteria

We considered three different error criteria

-

$$E^{(abs)} = \|\lambda - \tilde{\lambda}\|_\infty = \max_i \left\{ |\lambda_i - \tilde{\lambda}_i| \right\},$$

-

$$E^{(rel)} = \max_i \left\{ \frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|} \right\}$$

-

$$E^{(rel2)} = \frac{\|\lambda - \tilde{\lambda}\|_\infty}{\|\lambda\|_\infty} = \frac{E^{(abs)}}{|\lambda_{\max}|}$$



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-

$$E^{(rel2)} = \frac{\|\lambda - \tilde{\lambda}\|_\infty}{\|\lambda\|_\infty} = \frac{E^{(abs)}}{|\lambda_{\max}|}$$



Test suite

- Type I: Chebyshev–Comrade matrices, $A = T_n + \mathbf{u}\mathbf{e}_n^T$,

$$T_n = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & & \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ & & & & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Are involved in the computation of the roots of a polynomial expressed in the first kind Chebyshev polynomial basis



Chebyshev-Comrade matrices

$\mathbf{u} = \alpha [1, \dots, 1]^T$, for increasing values of α .

Table: Errors for Type-I matrices of size $n = 128$, and $\mathbf{u} = \alpha [1, \dots, 1]^T$.

α	1	10^3	10^5	10^7	10^8	10^{11}
$E^{(abs)}$	3.0631e-13	8.6616e-12	1.4552e-10	5.5879e-09	8.9407e-08	1.7243e-06
$E^{(rel)}$	3.0631e-13	8.6722e-12	8.5210e-11	5.5943e-09	1.4885e-08	3.8406e-06
$E^{(rel2)}$	1.6396e-13	8.6555e-15	1.4552e-15	5.5879e-16	8.9407e-16	1.7243e-17
avrgit	2.6371	2.8182	2.8099	2.8099	2.7934	3.1736



Chebyshev-Comrade matrices

Table: Errors for Type-I matrices of the form $T_n + \mathbf{u}\mathbf{e}_n^T$, with $\mathbf{u} = \text{rand}(n, 1)$. For different values of n , the absolute and relative errors are shown.

n	$E^{(abs)}$	$E^{(rel)}$	$E^{(rel2)}$
50	1.0947e-13	1.0947e-13	8.6909e-14
100	4.2141e-13	2.4800e-11	3.5758e-13
150	3.4097e-12	2.1823e-12	2.1823e-12
200	3.6731e-12	2.6981e-12	2.6981e-12
300	6.8008e-12	5.0597e-12	5.0597e-12
500	1.7089e-11	1.3405e-11	1.3405e-11



Test suite

- Type-II: Random tridiagonal plus random rank-one permutation. $A = T + \mathbf{u}\mathbf{e}_n^T$, where T is a random symmetric tridiagonal matrix, and \mathbf{u} is a random vector in $[0, 1]$.

Table: Errors for Type-II matrices, and for different values of n . The values reported represent the average error obtained over 50 randomly generated instances.

n	$E^{(abs)}$	$E^{(rel)}$	$E^{(rel/2)}$
100	2.2438e-12	1.3879e-12	1.0361e-12
150	3.4440e-10	5.6035e-10	1.5864e-10
200	5.7883e-12	4.9680e-12	2.6019e-12
300	1.5570e-12	7.6508e-12	7.2006e-13
500	4.2696e-12	3.0138e-11	1.8976e-12



Test suite

- Type-III: Unsymmetric tridiagonal matrices. We consider an almost symmetric tridiagonal matrix of the form

$$T = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 0 & 1 \\ & & & \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 0 & \alpha \\ & & & \alpha & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 - \alpha \end{bmatrix} \mathbf{e}_n^T. \quad (1)$$

When α becomes very large, the dominant eigenvalues are ill-conditioned.



Test suite

- Type-III: Unsymmetric tridiagonal matrices.

Table: Errors for Type-III matrices, $n=128$, and different values of α .

α	1	10	10^2	10^3	10^5	10^7	10^8
$E^{(abs)}$	5.8842e-14	7.9892e-13	9.8765e-13	1.6662e-12	5.8321e-11	3.1127e-09	1.8700e-08
$E^{(rel)}$	5.8842e-14	4.9782e-13	4.3876e-13	3.1974e-13	5.1728e-11	1.0268e-09	9.2553e-09
$E^{(rel2)}$	2.9430e-14	2.3967e-13	9.8270e-14	5.2664e-14	1.8443e-13	9.8434e-13	1.8700e-12
avrgit	2.9098	2.9268	3.3719	3.3033	2.9016	3.0656	2.9431



Test suite

- Type-IV: Hamiltonian-like matrices.

$$A = \begin{bmatrix} D & 0 \\ 0 & -\Delta \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{e}^T, -\mathbf{q}^T \end{bmatrix},$$

Table: Errors for Type-IV matrices of different size. The results are obtained for $\alpha = 0.9$ and $\beta = 0.8$.

n	$E^{(abs)}$	$E^{(rel)}$	$E^{(rel/2)}$
50	3.1335e-12	1.8983e-12	9.6708e-15
100	6.8070e-12	3.4106e-12	1.0694e-14
150	1.0714e-11	1.0714e-11	1.1290e-14
200	1.8701e-11	1.8701e-11	1.4825e-14
500	7.2838e-11	7.2838e-11	2.3223e-14
1000	8.0850e-10	8.0850e-10	1.2912e-13



Test suite

- Type-V: Hessenberg form of random matrices.

Table: Type-V: Hessenberg form of randomly generated matrices.

n	$E^{(abs)}$	$E^{(rel)}$	$E^{(rel/2)}$
50	2.0783e-12	2.0783e-12	3.2932e-14
100	9.5799e-12	9.5799e-12	7.6777e-14
150	5.5904e-11	6.2688e-12	2.9514e-13
200	8.2377e-11	8.2377e-11	3.2935e-13
500	9.1855e-11	3.2461e-11	1.4710e-13
1000	3.4718e-09	5.1914e-10	2.7480e-12



Conclusions and future work

- A new implicit- QR algorithm on rank-structured matrices
 - Givens-weight representation is the keyword
 - A new deflation technique
 - Effectiveness of the representation on iteration schemes
-
- Backward error analysis
 - Get rid of the explicit computations!



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