Epistemic Logic for Security

Lorenzo Ceragioli

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Epistemic Logic

Hypothesis (Safe approximations)

- Doxastic Logic, logic of belief
  V.S
  Epistemic Logic, logic of knowledge
- Everything agents knows is true
- Logical Omniscience
  - perfect reasoner
  - no awareness problems

Language

Given

At	set of atomic propositions
Ag	set of agent symbols
Op	set of modal operators

where usually *Op* depends on *Ag*.

$\varphi \in$ ***L(At, Op, Ag)*** defined by *BNF*

$$\varphi \; = \; p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi$$

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Most simple language: ***Op*** $= \{K_a \mid a \in \mathit{Ag}\}$

Only Simple Knowledge Language

$$\begin{aligned} \mathsf{At} &= \{S_a \mid a \in \mathit{Ag} \wedge S \in \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}\} \\ \mathsf{Ag} &= \{A, B, C\} \text{ for Alice, Bob and Charlie} \\ \mathsf{Op} &= \{K_a \mid a \in \mathit{Ag}\} \end{aligned}$$

In ***L(At, Op , Ag)*** we can say

$\heartsuit_B$	Bob has a hearth card
$K_A \spadesuit_A$	Alice knows that she has a spade card
$\neg K_B \heartsuit_B$	Bob doesn't know he has a hearth card
$\neg K_B K_A \spadesuit_A$	Bob doesn't know that Alice knows his card is spade
$\diamondsuit_C \wedge K_C \clubsuit_C$	Charlie has a diamond card but knows (know, not believe!) to have a club one

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R : $Ag \rightarrow W \times W$ is a function yielding an accessibility relation R_a for each agent a, ideally the world w' is accessible from w using R_a (we will write $w \xrightarrow{a} w'$) if in world w the agent a think w' to be possible given its knowledge

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V $: W \to (At \to \{true, false\})$ is a function that for each world w yield a propositional valuation $V(w)$ such that $V(w)(p) \mapsto true$ iff p is true in world w

Truth in a Kripke Model

Given a Kripke Model $M = (W,R,V)$ and a world w we define what it means for a formula φ to be true in (M, w), written $M, w \vDash \varphi$

$M, w \vDash p$	iff $V(w)(p) = true$ where $p \in At$
$M, w \vDash \varphi \wedge \psi$	iff $M, w \vDash \varphi$ and $M, w \vDash \psi$
$M, w \vDash \varphi \vee \psi$	iff $M, w \vDash \varphi$ or $M, w \vDash \psi$
$M, w \vDash \neg\varphi$	iff $M, w \nvDash \varphi$
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$M, w \vDash \varphi \wedge \psi$	iff $M, w \vDash \varphi$ and $M, w \vDash \psi$
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$M, w \vDash \neg\varphi$	iff $M, w \nvDash \varphi$
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And we write

$$M \vDash \varphi \qquad \text{iff } M, w \vDash \varphi \text{ for all } w \in W$$

Example

Suppose to have two players Alice and Bob with a deck of three cards $\heartsuit$, $\spadesuit$, $\clubsuit$. Each player pick a card, each player knows only his card and the rules of the game.

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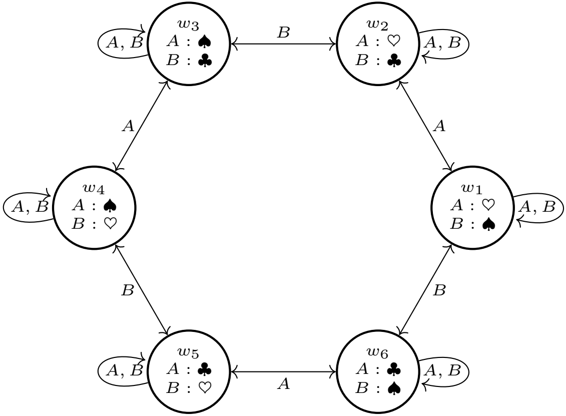

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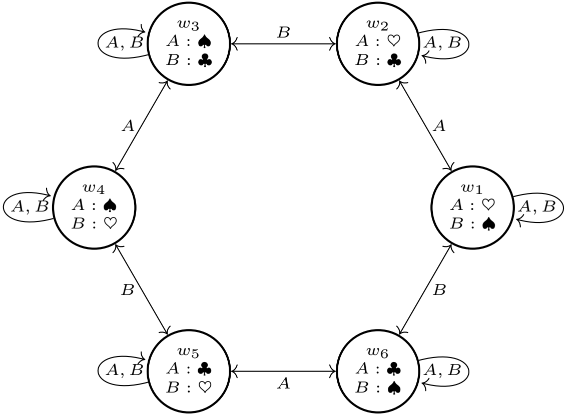

$$M, w_1 \vDash \heartsuit_A \wedge K_A \heartsuit_A$$

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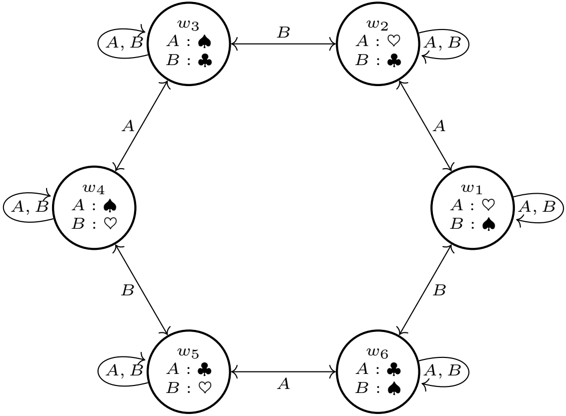

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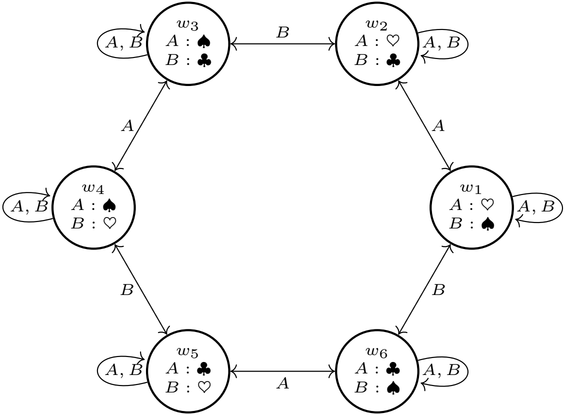

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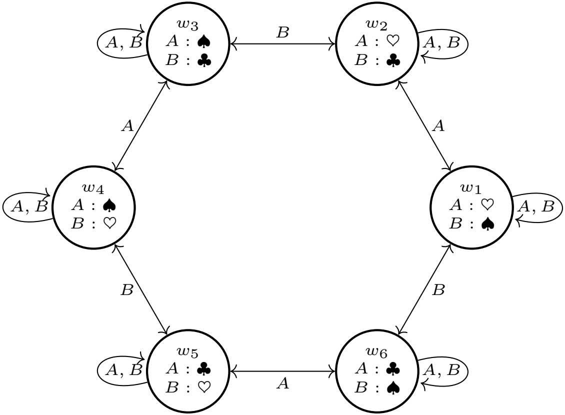

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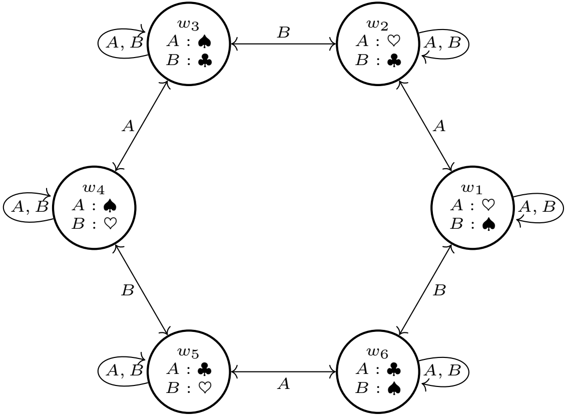

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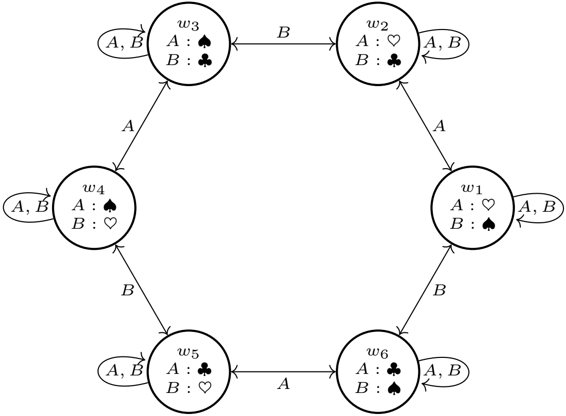

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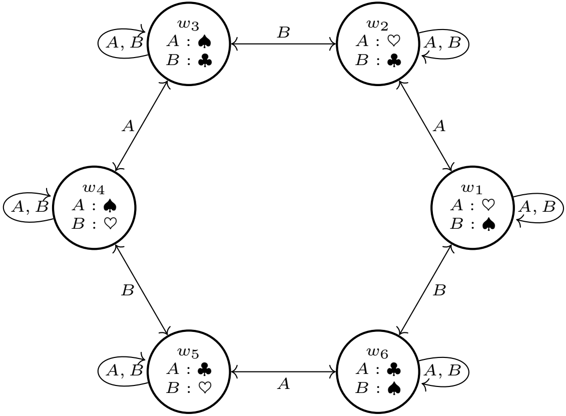

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- Classes of models

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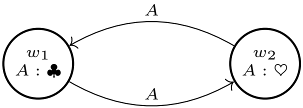

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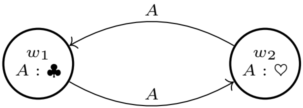

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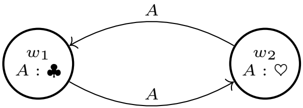

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- $\mathcal{S}5 \subseteq \mathcal{K}$ class of models

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negative introspection: $\mathcal{S}5 \models \neg K_a\varphi \rightarrow K_a \neg K_a\varphi$

Not as simple as you could believe now

- It's not always the case that there is only one possible world for each combination of truth value for atomic predicates
- This epistemic logic is propositional, but we can consider a predicative version (also first order)
- There can be an infinite number of possible worlds (especially in the predicative case)

Example

We have a deck with two cards $\heartsuit$ and $\spadesuit$, we take one card and put it covered on the table. We have two players, Alice and Bob, Alice can cheat and look at the card, Bob cannot. Bob knows that Alice can cheat, but if she does he wouldn't know.

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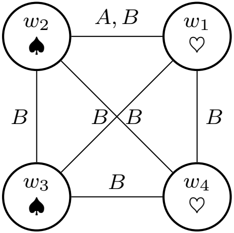

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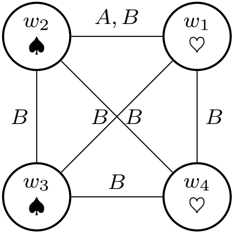

$$M \vDash \neg K_B \neg K_A \spadesuit$$

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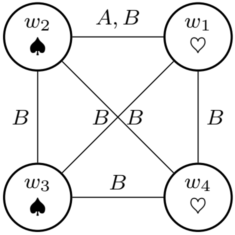

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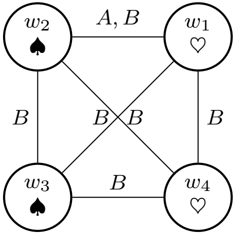

$$M \vDash \neg K_B \neg K_A ♠$$

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$$M \vDash \neg K_B (K_A ♡ \vee K_A ♠)$$

Beyond Simple Knowledge: Group Knowledge

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- Common Knowledge of φ among $G \subseteq Ag$, we write $C_G\varphi$

Semantics of Group Knowledge

$M, w \vDash E_A\varphi$ iff for all w' such that $(w, w') \in R_{E_A}$, we have $M, w' \vDash \varphi$

where $R_{E_A} = \bigcup_{a \in A} R_a$

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$M, w \vDash C_A\varphi$ iff for all w' such that $(w, w') \in R_{C_A}$, we have $M, w' \vDash \varphi$
where $R_{C_A} = (\bigcup_{a \in A} R_a)^+$

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- $\mathcal{K} \vDash C_G\varphi \to E_G\varphi$

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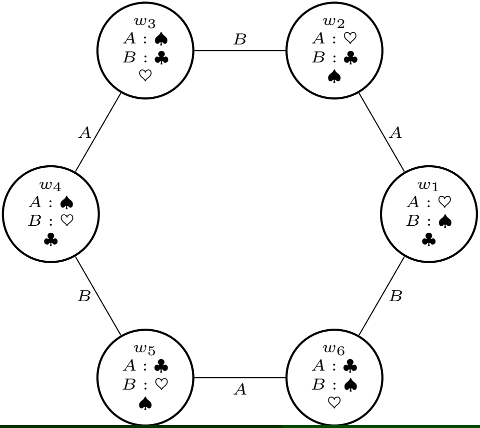

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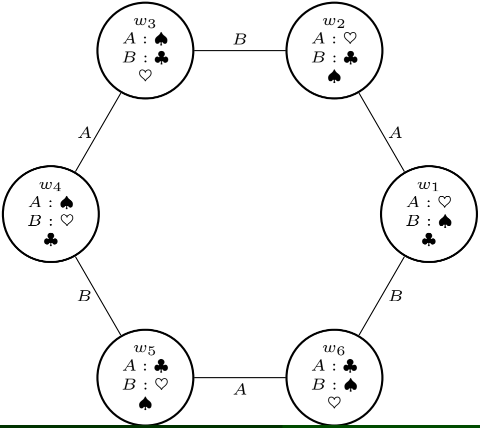

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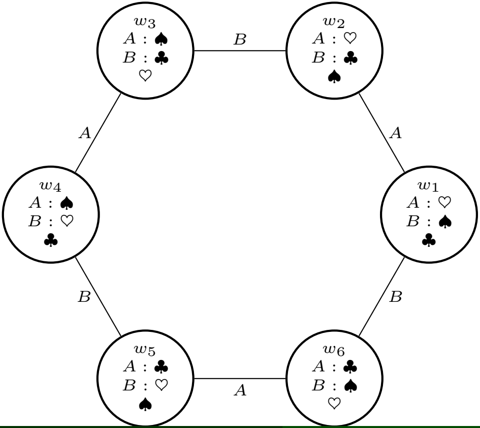

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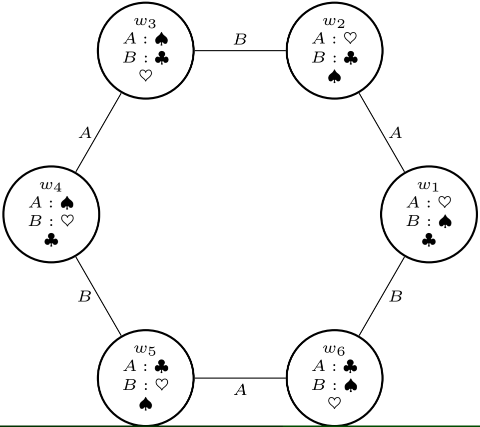

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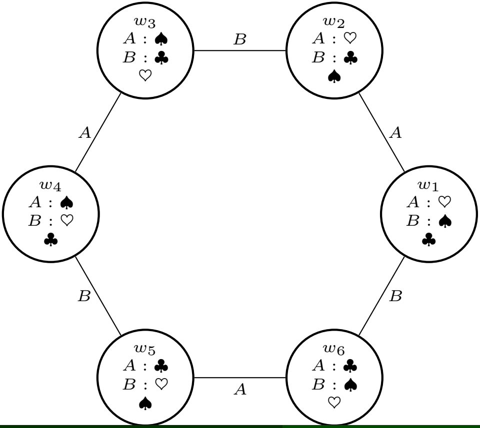

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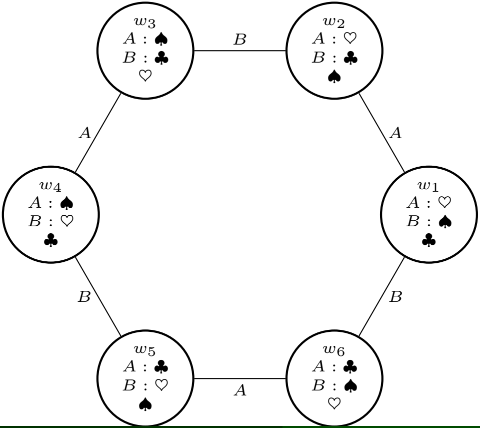

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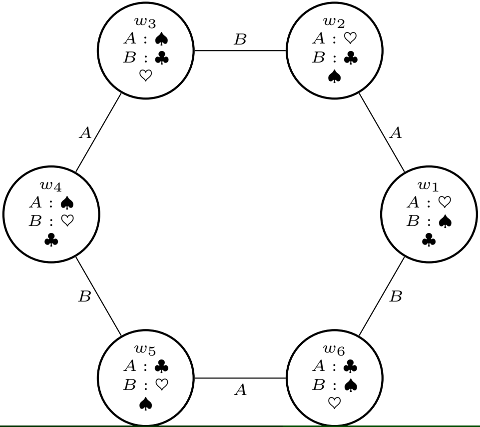

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The first deck, A, contains ■, ■ and ■.
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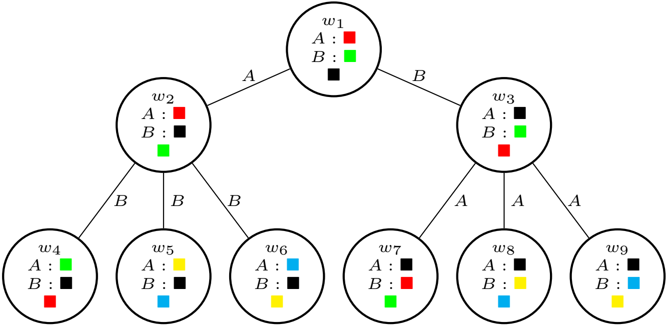

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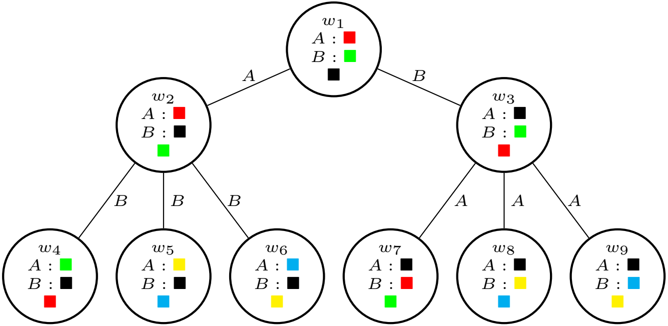
$$M, w_1 \models K_A(\neg {\color{cyan}\blacksquare})$$



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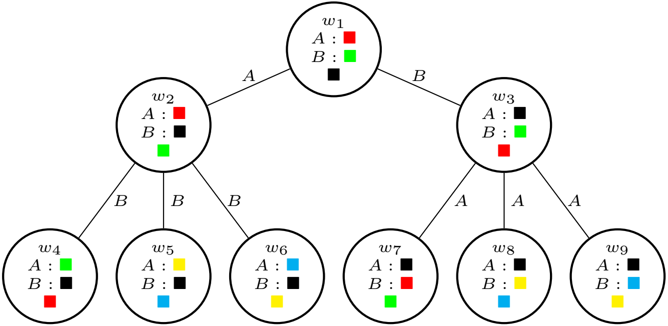

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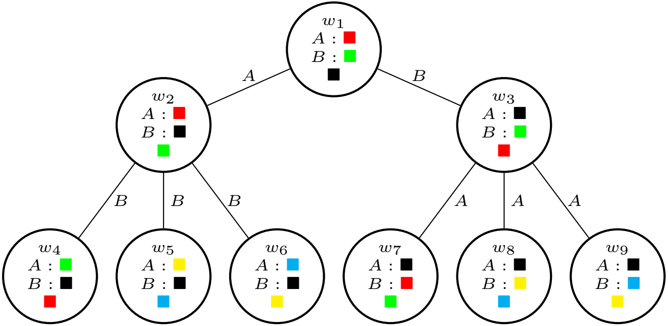

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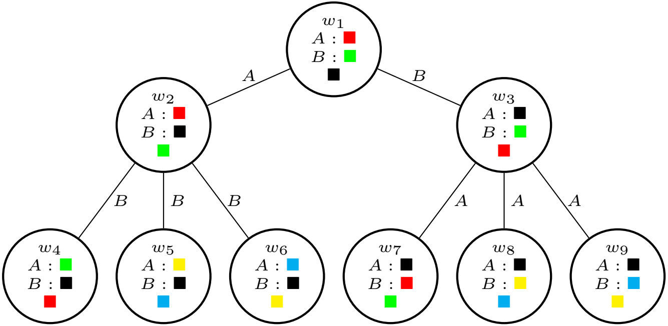

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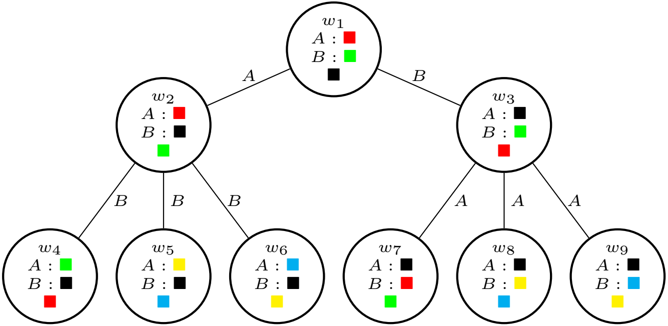

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- Knowledge of agents
- Knowledge of groups of agents

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Then Alice reach the table, look the card and let it covered on the table, all in front of Bob.

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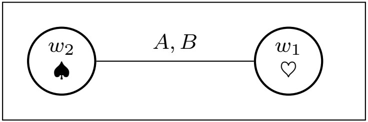

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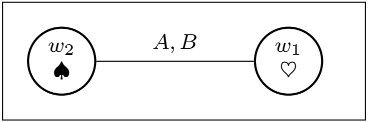

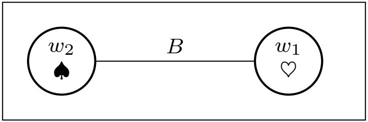

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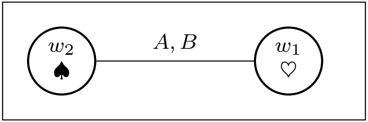

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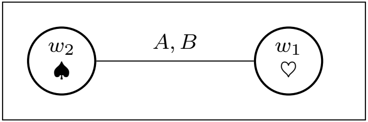

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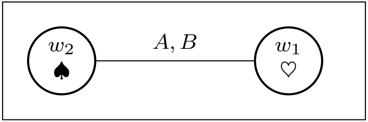

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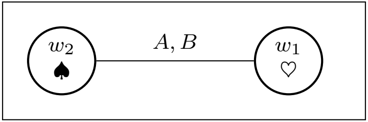

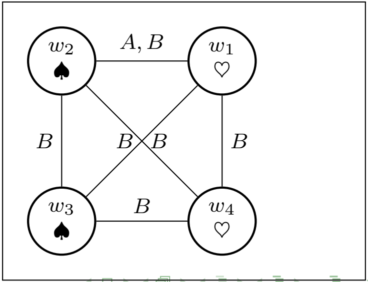

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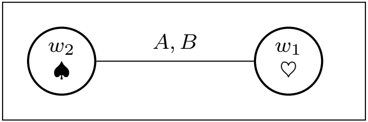

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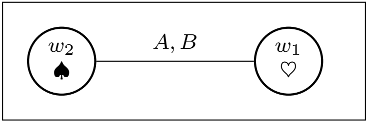

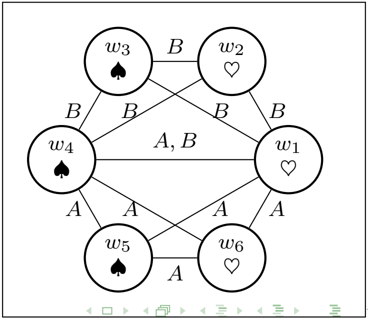

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E.g. Public announcement

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- pre : $S \to \mathcal{L}$ is a precondition function that assign a precondition $pre(\sigma) \in \mathcal{L}$ to each $\sigma \in S$

An **epistemic action** is a pointed action model (U, σ) with $\sigma \in S$.

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- The valuation function is such that $V(w, \alpha) = V(w)$

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where $p \in$ *At*, $\Box \in$ *Op* (probably K_a, C_a, E_a, D_a for $a \in Ag$) and $\alpha \in S$.

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Formally

$$M, w \vDash [\alpha]\varphi \;\text{ iff }\; M, w \vDash pre(\alpha) \text{ implies } M \otimes U, (w, \alpha) \vDash \varphi$$

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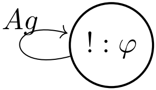
Action model for the announcement of $\varphi \in \mathcal{L}$



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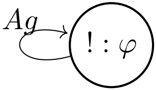

The result of applying this action is simply to remove the possible worlds in which ϕ doesn't hold (**relativization of the model** M **in** M_φ).

Example: muddy children

Three children Alice, Bob and Charlie play in the garden, their mother told them to not get dirty. No one of the children can see if he is dirty on the back or not, but everyone can see all his siblings. Then initially every child knows only the state of the other children.

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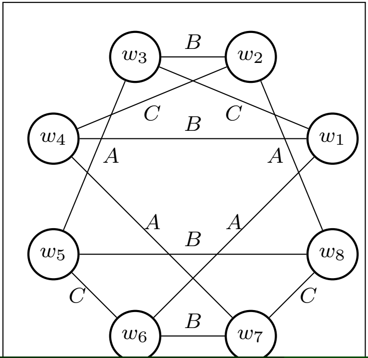

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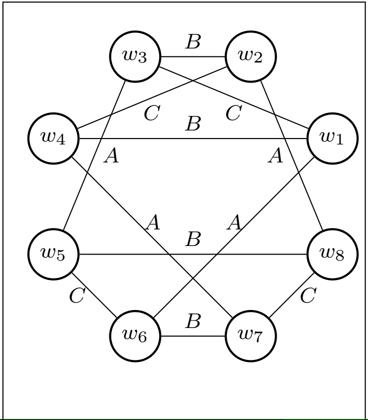

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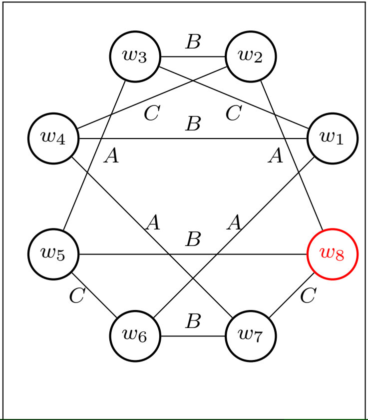

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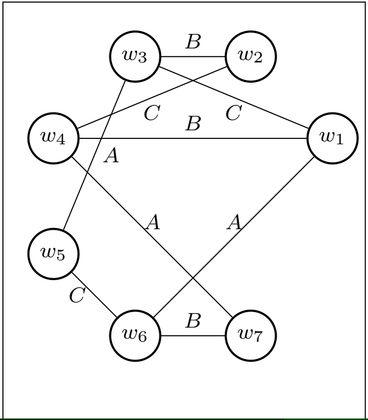

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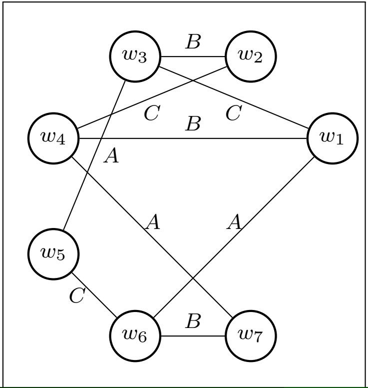

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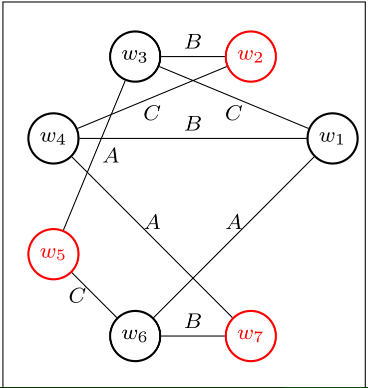

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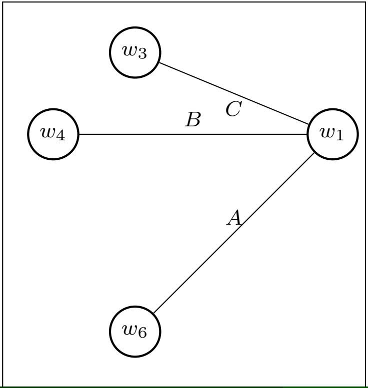

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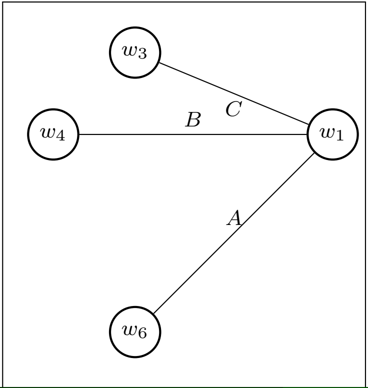

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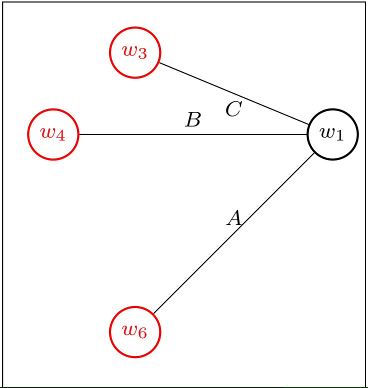

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Now each child knows that he and his siblings are all dirty!

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Then we can say that

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$$M \vDash [!\varphi_1][!\varphi_2][!\varphi_2](K_A D_A \wedge K_B D_B \wedge K_C D_C)$$

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E.g. (from the Moore's paradox) $(p \wedge \neg K(p))$ is unsuccessful

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But we can state the fact that:
If φ is successful ($\mathcal{K} \vDash [!\varphi]\varphi$) then also $\mathcal{K} \vDash [!\varphi]C\varphi$

Epistemic Logic for Security

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- A lot (not all) security properties specify what each agent must know and mustn't know
  - confidentiality
  - agent authentication
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- Ideally epistemic logic should be a good tool for speaking about security
  - formal proof systems
  - higher level analysis (better then bisimulation)
  - knowledge and ability
  - knowledge and time
  - knowledge and strategy (epistemic foundation of game theory)

Uses

There are two possible uses of epistemic logic in security

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Still fewer successes...

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We already know BAN logic

Spatial-Epistemic Logic

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- process calculus based on π-calculus, but with lots of freedom on messages (customization)
- automatic derivation of *Dolev-Yao* attacker

Model

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- To specify the model we need a Signature Σ and a set of equations E

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- The set of terms $s, t, v \in Terms =$
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- An equational theory E to define the semantics of function symbols in Σ

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- We will assume that E is subterm convergent

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Given Σ and E we define

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Constructor: Any other function symbol in Σ

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- $E = \{dec(enc(x, y), y) \to x\}$
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If $\varphi \in \mathcal{F}(\psi)$ then we write $\psi \Vdash \varphi$ (Knowledge derivation)

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We have that $\{k_2, enc(k_1, k_2), enc(m, k_1)\} \Vdash m$

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- $\{k_2, enc(k_1, k_2), enc(m, k_1)\} \subseteq \mathcal{F}(\psi)$
- $\{k_1\} \in \mathcal{F}(\psi)$ since $dec \in$ destructor(E) and $enc(k_1, k_2) \in \mathcal{F}(\psi)$ and $k_2 \in \mathcal{F}(\psi)$ and $dec(enc(k_1, k_2), k_2) \to k_1 \in E$

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- $\{m\} \in \mathcal{F}(\psi)$ since $dec \in$ destructor(E) and $enc(m, k_1) \in \mathcal{F}(\psi)$ and $k_1 \in \mathcal{F}(\psi)$ and $dec(enc(m, k_1), k_1) \to m \in E$

Process Calculus

Processes P, Q are defined in BNF

$$
\begin{aligned}
P, Q ::= {} & 0 && \textit{(Null Process)} \\
\mid {} & P|Q && \textit{(Parallel Composition)} \\
\mid {} & (\nu n)P && \textit{(Name Restriction)} \\
\mid {} & \alpha.P && \textit{(Action Prefix)} \\
\mid {} & P + Q && \textit{(Choice)} \\
\mid {} & \texttt{let } x = T \texttt{ in } P && \textit{(Let Construct)}
\end{aligned}
$$

Process Calculus

$$
\begin{aligned}
\alpha ::= {} & m(x) && \textit{(Input)} \\
\mid {} & m\langle T \rangle && \textit{(Output)} \\
\mid {} & m\langle * \rangle && \textit{(Attacker Output)} \\
\mid {} & [T_1 = T_2] && \textit{(Test)}
\end{aligned}
$$

$$
\begin{aligned}
T ::= {} & n && \textit{(Name)} \\
\mid {} & x && \textit{(Variable)} \\
\mid {} & f(T_1, \ldots, T_n) && \textit{(Function)}
\end{aligned}
$$

Structural Congruence

if $n \notin fn(P) \cup fv(P)$ then $P|(\nu n)Q \equiv (\nu n)(P|Q)$

$(\nu n)0 \equiv 0$

$(\nu n)(\nu m)P \equiv (\nu m)(\nu n)P$

if $M_1 =_E M_1'$ then `let x =` M_1 `in` $P \equiv$ `let x =` M_1' `in` P

if $M_1 =_E M_1'$ then $m\langle M_1 \rangle.P \equiv m\langle M_1' \rangle.P$

if $M_1 =_E M_1'$ then $[M_1 = M].P \equiv [M_1' = M].P$

$P|0 \equiv P$

$P|Q \equiv Q|P$

$P|(Q|R) \equiv (P|Q)|R$

$P + Q \equiv Q + P$

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Reduction Semantics

Computational steps **inside** a process (no interaction with the environment)

$$\textit{(Let)}\ \frac{M \text{ is destructor free}}{\texttt{let x = } M \texttt{ in } P \longrightarrow P\{x \leftarrow M\}}$$

$$\textit{(Sync)}\ \frac{M \text{ is destructor free}}{n\langle M\rangle.P + R \mid n(x).Q + S \longrightarrow P \mid Q\{x \leftarrow M\}}$$

$$\textit{(Test)}\ \frac{M_1 \text{ and } M_2 \text{ is destructor free} \qquad M_1 =_E M_2}{[M_1 = M_2].P \longrightarrow P}$$

Reduction Semantics

$$\textit{(Par)}\ \frac{P \to Q}{P|R \to Q|R} \qquad \textit{(Scope)}\ \frac{P \to Q}{(\nu n)P \to (\nu n)Q}$$

$$\textit{(Cong)}\ \frac{P \equiv P' \qquad P' \to Q' \qquad Q \equiv Q'}{P \to Q}$$

$$\textit{(Attacker)}\ \frac{M \in \mathcal{F}(gt(Q) \cup \bar{n}) \text{ with } \bar{n} \text{ fresh names}}{c(x).P + R \mid c\langle * \rangle.Q + S \longrightarrow (\nu \bar{n})(P\{x \leftarrow M\}|Q)}$$

Labelled Transition Semantics

Communication with the **environment**

$$\textit{(Tau)}\ \frac{P \to Q}{P \xrightarrow{\tau} Q}$$

$$\textit{(Out)}\ \frac{M \text{ is destructor free}}{n\langle M\rangle.P \xrightarrow{n\langle M\rangle} P}$$

$$\textit{(Inp)}\ \frac{M \text{ is destructor free}}{n(x).P \xrightarrow{n(M)} P}$$

$$\textit{(Attacker Out)}\ \frac{M \in \mathcal{F}(gt(Q) \cup \bar{n}) \text{ with } \bar{n} \text{ fresh names}}{c\langle *\rangle.P \xrightarrow{\nu\bar{n}.c\langle M\rangle} P}$$

Labelled Transition Semantics

$$\textit{(Res)}\ \frac{P \xrightarrow{\alpha} Q \qquad \forall n \in \bar{u}.n \notin \mathsf{names}\alpha}{(\nu\bar{u})P \xrightarrow{\alpha} (\nu\bar{u})Q}$$

$$\textit{(Bound Out)}\ \frac{P \xrightarrow{n\langle M\rangle} P' \qquad \bar{s} \subseteq \mathsf{names}(M) \text{ and } \bar{s} \subseteq \bar{u} \qquad \bar{u'} = \bar{u} \setminus \bar{s}}{(\nu\bar{u})P \xrightarrow{\nu\bar{s}.n\langle M\rangle} (\nu\bar{u'})P'}$$

$$\textit{(Cong)}\ \frac{P \equiv P' \qquad P' \xrightarrow{\alpha} Q' \qquad Q \equiv Q'}{P \xrightarrow{\alpha} Q}$$

Logic

Processes P, Q are defined in BNF

$$
\begin{aligned}
A, B ::= {} & T && \textit{(True)} \\
\mid {} & \neg A && \textit{(Negation)} \\
\mid {} & A \wedge B && \textit{(Conjunction)} \\
\mid {} & 0 && \textit{(Void)} \\
\mid {} & A|B && \textit{(Composition)} \\
\mid {} & H_x.A && \textit{(Hidden Quantification)} \\
\mid {} & \alpha.A && \textit{(Action)} \\
\mid {} & \Box A && \textit{(Always)} \\
\mid {} & \Diamond A && \textit{(Eventually)} \\
\mid {} & @n && \textit{(free name predicate)} \\
\mid {} & K\varphi && \textit{(Knowledge)} \\
\mid {} & Sx.A && \textit{(Secret Quantification)}
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi, \psi ::= {} & \varphi \wedge \psi && \textit{(Conjunction)} \\
\mid {} & t && \textit{(Term)} \\
\mid {} & \top && \textit{(True)}
\end{aligned}
$$

Semantics

$$
\begin{aligned}
&P \vDash T && \text{iff True}\\
&P \vDash \neg A && \text{iff not } \vDash A\\
&P \vDash A \wedge B && \text{iff } P \vDash A \text{ and } P \vDash B\\
&P \vDash 0 && \text{iff } P \equiv 0\\
&P \vDash A|B && \text{iff } \exists Q, R.\ P \equiv Q|R \text{ and } Q \vDash A \text{ and } R \vDash B\\
&P \vDash H_x.A && \text{iff } \exists Q.\ P \equiv (\nu n)Q \text{ and } Q \vDash A\{x \leftarrow n\}\\
&P \vDash \alpha.A && \text{iff } \exists Q.\ P \xrightarrow{\alpha} Q \text{ and } Q \vDash A
\end{aligned}
$$

Semantics

$$
\begin{aligned}
P &\vDash \Box A && \text{iff } \forall Q.\ P \xrightarrow{\tau}^{*} Q \text{ then } Q \vDash A \\
P &\vDash \Diamond A && \text{iff } \exists Q.\ P \xrightarrow{\tau}^{*} Q \text{ and } Q \vDash A \\
P &\vDash @n && \text{iff } n \in fn(P) \\
P &\vDash K\varphi && \text{iff } P \vdash_k \psi \text{ and } \psi \Vdash \varphi \\
P &\vDash Sx.A && \text{iff } \exists Q, t\ .P \equiv (\nu k)Q \text{ and } Q \vDash A\{x \leftarrow t\} \\
& && \text{and } Q \vdash_k \psi \text{ such that } t \in \psi \text{ and } k \in names(t)
\end{aligned}
$$

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  - Avoid to take terms that are not closed,
- We have to deal with restricted names (we use $\uparrow$)

Relevant Terms for a Process

$$\frac{}{0 \vdash_k \emptyset}$$

$$\frac{P \vdash_k \varphi \qquad Q \vdash_k \psi}{P + Q \vdash_k \varphi \cup \psi}$$

$$\frac{P \vdash_k \varphi \qquad Q \vdash_k \psi}{P|Q \vdash_k \varphi \cup \psi}$$

$$\frac{P \vdash_k \varphi}{n(x).P \vdash_k \varphi}$$

$$\frac{P \vdash_k \varphi}{x\langle M\rangle.P \vdash_k \varphi \cup sub(M)}$$

Relevant Terms for a Process

$$\frac{P\{x \leftarrow M\} \vdash_k \varphi}{\texttt{let } n = M \texttt{ in } P \vdash_k \varphi \cup sub(M)}$$

$$\frac{P \vdash_k \varphi}{(\nu n)P \vdash_k \varphi \uparrow n}$$

$$\frac{P \vdash_k \varphi}{[M = N].P \vdash_k \varphi \cup sub(M) \cup sub(N)}$$

Where: $\psi \uparrow x = \{t \mid t \in \psi \wedge x \notin names(t)\}$

Relevant Subterms of a Term

$$\frac{n \text{ is a name}}{sub(n) = \{n\}} \qquad \frac{x \text{ is a variable}}{sub(x) = \emptyset}$$

$$\frac{f \text{ is a destructor}}{sub(f(t_1, \ldots, t_n)) = sub(t_1) \cup \cdots \cup sub(t_n)}$$

$$\frac{f \text{ is a constructor} \qquad \nexists t_i \text{ with a variable or a destructor}}{sub(f(t_1, \ldots, t_n)) = \{f(t_1, \ldots, t_n)\}}$$

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Example: Knowledge

$\Sigma = \{enc/2, dec/2\}$ destructors$(E) = \{dec\}$
$E = \{dec(enc(x, y), y) \to x\}$ constructors$(E) = \{enc\}$
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and $\{k, enc(m, k)\} \Vdash m$

Example: Space

Number of parallel threads

$0 := 0$
$1 := \neg 0 \wedge \neg(\neg 0 \mid \neg 0)$
$2 := \neg 0 \wedge \neg 1 \wedge \neg(\neg 0 \mid \neg 0 \mid \neg 0)$

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$c\langle dec(k)\rangle.0 \mid c\langle enc(m,k)\rangle.0 \nvDash 1$

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The process has a restricted name and it is always the case that it send it through channel m
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$(\nu n).m\langle n\rangle.0 \mid (\nu c)(\nu n).c\langle m\rangle.m\langle n\rangle.0 \vDash (P \mid P)$

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$\Sigma = \{enc/2, dec/2\}$
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$(\nu k).tag\langle enc(k, c)\rangle.c\langle c\rangle.0 \mid (\nu k).tag\langle k\rangle.0 \vDash P$

$(\nu k).tag\langle enc(k, c)\rangle.c\langle c\rangle.0 \; + \; (\nu k).tag\langle k\rangle.0 \vDash P$

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$(\nu k).tag\langle enc(k, c)\rangle.c\langle c\rangle.0 \mid (\nu k).tag\langle k\rangle.0 \vDash P$

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- the idea is to express a procedure to create an extra process to put in parallel $P \mid E$

Attacker

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- then we have to prove that the process $P \mid E$ satisfy the logical formula asserting the property of interest

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Procedure for generating attacker:

```
proc Attacker(P,S){
    if (P -α-> Q ∧ α = input on c) then c⟨∗⟩.Attacker(Q,S)
    if (P -α-> Q ∧ α = output on c) then c(x).Attacker(Q,S∪{x})
    if ( P -α-/-> ) then m⟨x_1,...,x_n⟩ where x_i ∈ S
```

$$\texttt{if } (P \xrightarrow{\alpha} Q \wedge \alpha = \texttt{input on } c) \texttt{ then } c\langle * \rangle\texttt{.Attacker(Q,S)}$$

$$\texttt{if } (P \xrightarrow{\alpha} Q \wedge \alpha = \texttt{output on } c) \texttt{ then } c(x)\texttt{.Attacker(Q,S}\cup\{x\}\texttt{)}$$

$$\texttt{if } (\ P \stackrel{\alpha}{\not\to}\ ) \texttt{ then } m\langle x_1, \ldots, x_n \rangle \texttt{ where } x_i \in S$$

Example

$\Sigma =$
$\{enc/2, dec/2, pair/2, \pi_1/1, \pi_2/1, t/1\}$

$$E = \{dec(enc(x, y), y) \to x$$
$$\pi_1(pair(x, y)) \to x$$
$$\pi_2(pair(x, y)) \to y\}$$

destructors$(E) =$
$\{dec, \pi_1, \pi_2\}$

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We consider the following trivial protocol

$$A \to B : \{key_{ab}, N\}_{key}$$
$$B \to A : \{N - 1\}_{key_{ab}}$$

Example

Protocol modeling:

$$
\begin{aligned}
A(key) &= (\nu key_{ab}, N) \\
&\qquad c\langle enc(pair(key_{ab}, N), key)\rangle. \\
&\qquad c(x).[t(N) = dec(x, key_{ab})].\ ok\langle ok\rangle \\
B(key) &= c(x).\mathtt{let} \\
&\qquad key_{ab} = \pi_1(dec(x, key)) \\
&\qquad N = \pi_2(dec(x, key)) \\
&\qquad\qquad \mathtt{in}\ \ c\langle enc(t(N), key_{ab})\rangle \\
Sys &= (\nu key)(A(key) \mid B(key))
\end{aligned}
$$

Example

Attacker modeling:

$$E = c(x).\ c\langle * \rangle.\ c(y).\ c\langle * \rangle.\ mem\langle c, y \rangle$$
$$World = (Sys \mid E)$$

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we can see:

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$$World \vDash \neg \Diamond Hkey.(ok(ok).T \wedge (1|\neg Kkey|@mem))$$

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