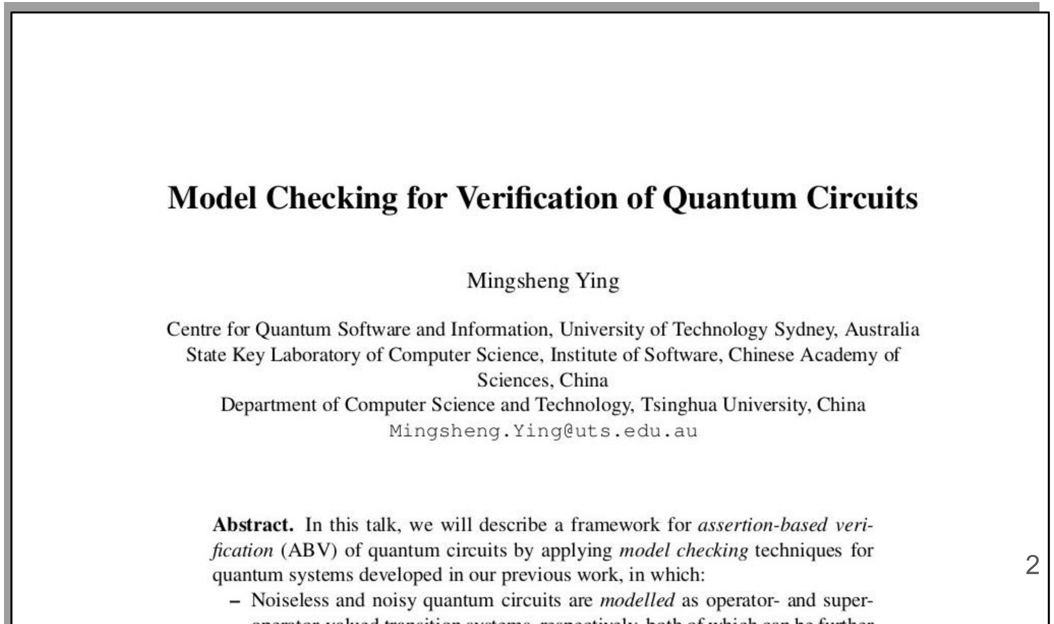
Model Checking Quantum Circuits

Approach

- Based on the paper by Ying

- A **pragmatic approach**

- We will give some context and reason about choices



Model Checking for Verification of Quantum Circuits

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Abstract. In this talk, we will describe a framework for *assertion-based verification* (ABV) of quantum circuits by applying *model checking* techniques for quantum systems developed in our previous work, in which:

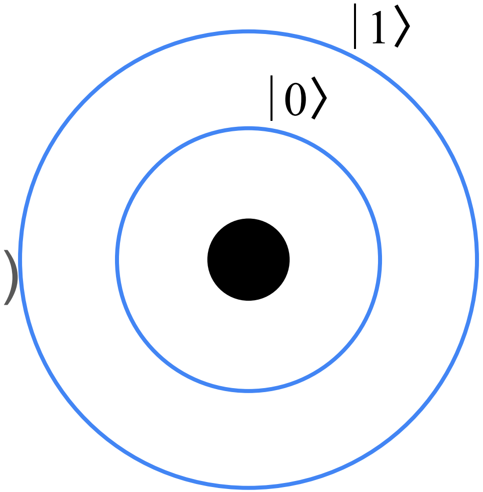
– Noiseless and noisy quantum circuits are *modelled* as operator- and super-

Content

- Brief **Introduction** to Quantum Circuits
- **Modeling** Quantum Circuits as Transition Systems
- A **Logic** for Temporal Properties on Quantum Systems
- Reduction to CTL **model checking**
- Dealing with **Mixed States**
- **Optimization** via Tensor Networks

Quantum Circuits

Qubits



- Two **classic states** $|0\rangle$ and $|1\rangle$
- Pure states (**superposition** of classic states)

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1$$

- **probability** of each classic state
- **wave** phase (interference)

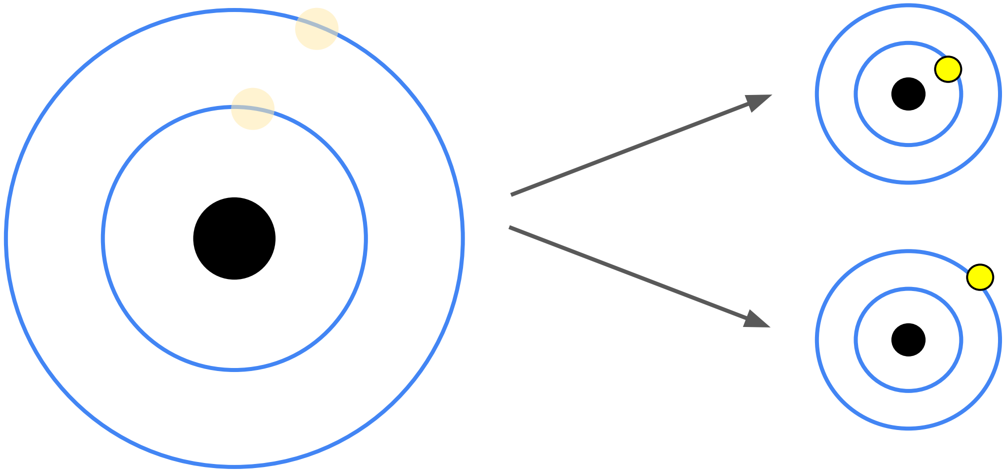
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Measurements

Outcome on $|\varphi\rangle = a|0\rangle + b|1\rangle$

- $|0\rangle$ with probability $|a|^2$
- $|1\rangle$ with probability $|b|^2$

The system **decays** in the observed classical state



$$|\varphi'\rangle = 1|0\rangle + 0|1\rangle$$

$$|\varphi''\rangle = 0|0\rangle + 1|1\rangle$$

Dynamics of an (isolated) quantum system

$|\varphi\rangle \rightarrow |\varphi'\rangle$ with $|\varphi'\rangle = U\,|\varphi\rangle$

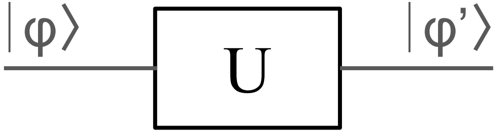
U is a transformation

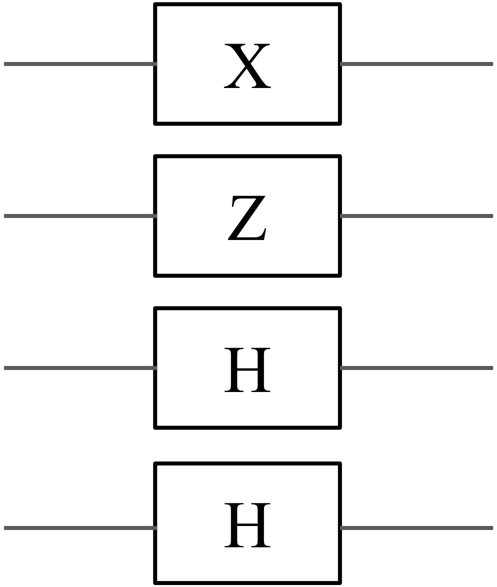
- **Linear**: $U(a|0\rangle + b|1\rangle) = a\,U|0\rangle + b\,U|1\rangle$
- **Unitary**: $U^\dagger U = U\,U^\dagger = I$

Single Qubit Transformations (Gates)

$|\varphi\rangle \longrightarrow |\varphi'\rangle$ with $|\varphi'\rangle = U\,|\varphi\rangle$

where $|\varphi\rangle = a|0\rangle + b|1\rangle$





$a|0\rangle + b|1\rangle$ — X — $b|0\rangle + a|1\rangle$

$a|0\rangle + b|1\rangle$ — Z — $a|0\rangle - b|1\rangle$

$|0\rangle$ — H — $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

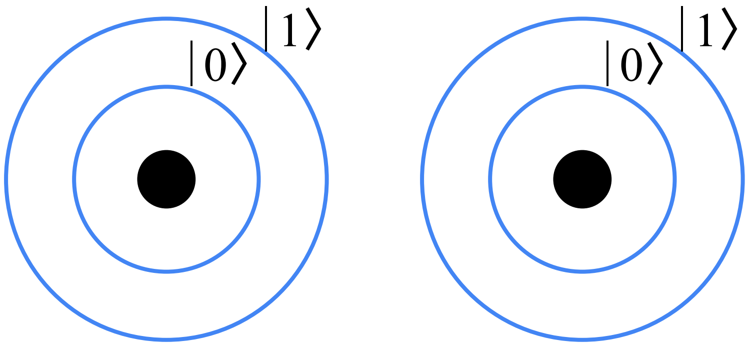
$|1\rangle$ — H — $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Multiple Qubits Systems

- **Classic states** $|00\rangle, |01\rangle, |10\rangle$ **and** $|11\rangle$
- **Pure states**

$$|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$a, b, c, d \in \mathbb{C} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$



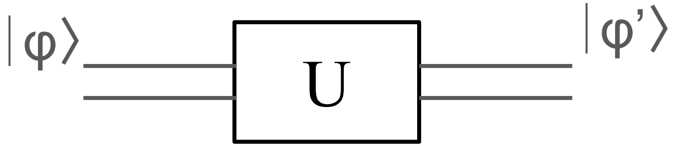
Entangled States

$$|\varphi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

- $|0\rangle$ or $|1\rangle$ with equal probability for both qubits
- BUT they **must be equal**
  - when one is measured, both of them decay

You **cannot decompose** the system in two components

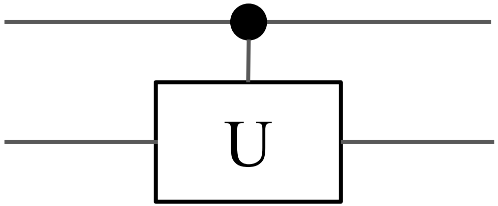
Two Qubits Transformations (Gates)



$|\varphi\rangle \longrightarrow |\varphi'\rangle$ with $|\varphi'\rangle = U\,|\varphi\rangle$
where $|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

Only a specific case

- **control** qubit
- **target** qubit

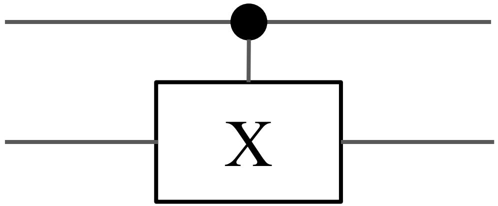


$|0,x\rangle \longrightarrow |0,x\rangle$

$|1,x\rangle \longrightarrow |1,Ux\rangle$

Controlled NOT

$|\varphi\rangle \longrightarrow |\varphi'\rangle$



$|0x\rangle \longrightarrow |0x\rangle$

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$

CNOT can create entanglement

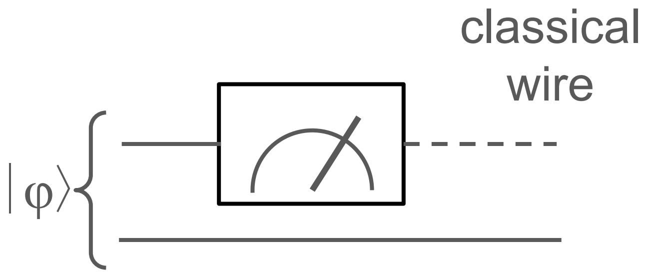
$|\varphi\rangle$ is non entangled $\quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$|\varphi'\rangle$ is entangled $\quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Measurements (Again)

Outcome of measuring **the first** qubit of
$|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

- $|0\rangle$ with probability $p(0) = |a|^2 + |b|^2$
- $|1\rangle$ with probability $p(1) = |c|^2 + |d|^2$

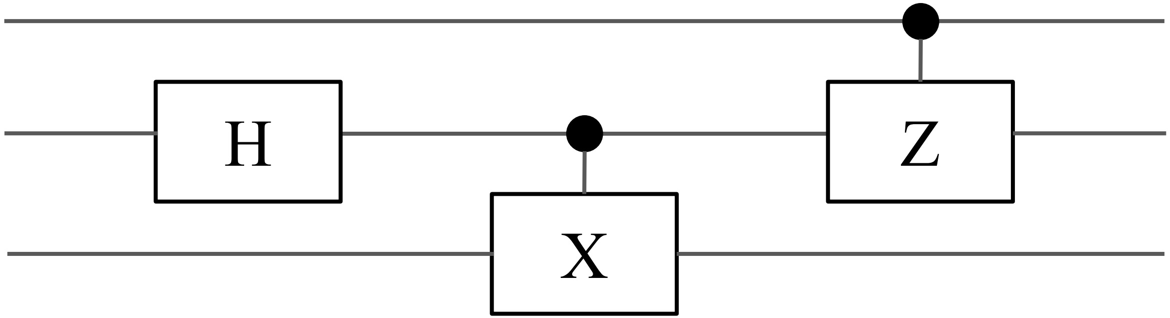


The system **decays** according to the observed classical state
Operators that applied to $|\varphi\rangle$ returns the new state:

- $M_{|0\rangle} / \sqrt{p(0)}$ where $M_{|0\rangle} = |0\rangle\langle 0|$ with $\langle 0| = (\ 1\ 0\ )$
- $M_{|1\rangle} / \sqrt{p(1)}$ where $M_{|1\rangle} = |1\rangle\langle 1|$ with $\langle 1| = (\ 0\ 1\ )$

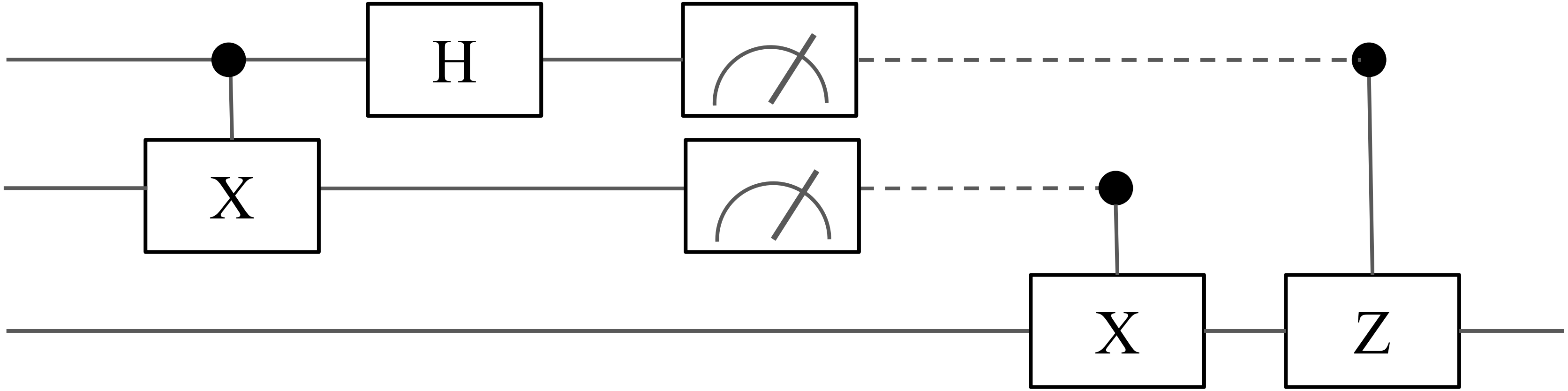
Combinatorial Quantum Circuits

Just a **composition** of gates on qubit wires
Measurements only at the end of computation



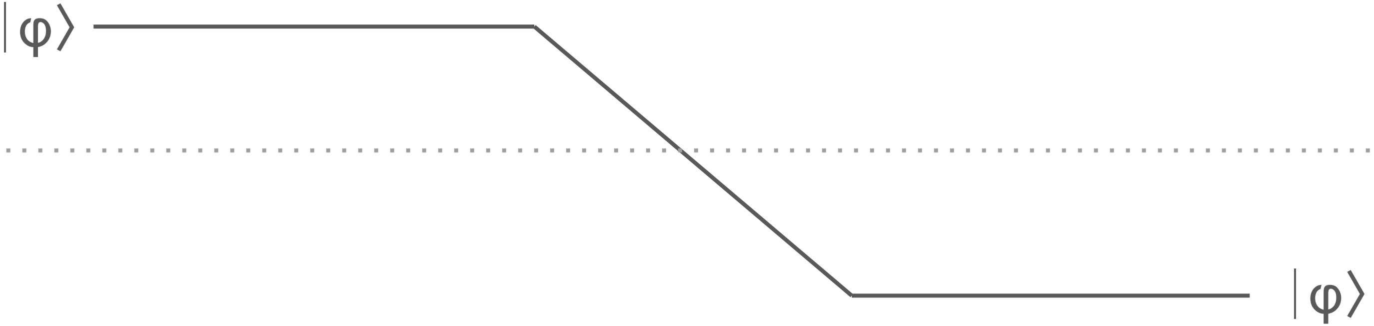
Dynamic Quantum Circuits

- **Quantum** bit wires
- **Classical** bit wires
- Quantum **Gates** (also controlled by classical bits)
- **Measurements** in arbitrary points of the computation



Quantum Teleportation

- Alice send $|\varphi\rangle$ to Bob using **classical communication**
- They can start with **entangled qubits**



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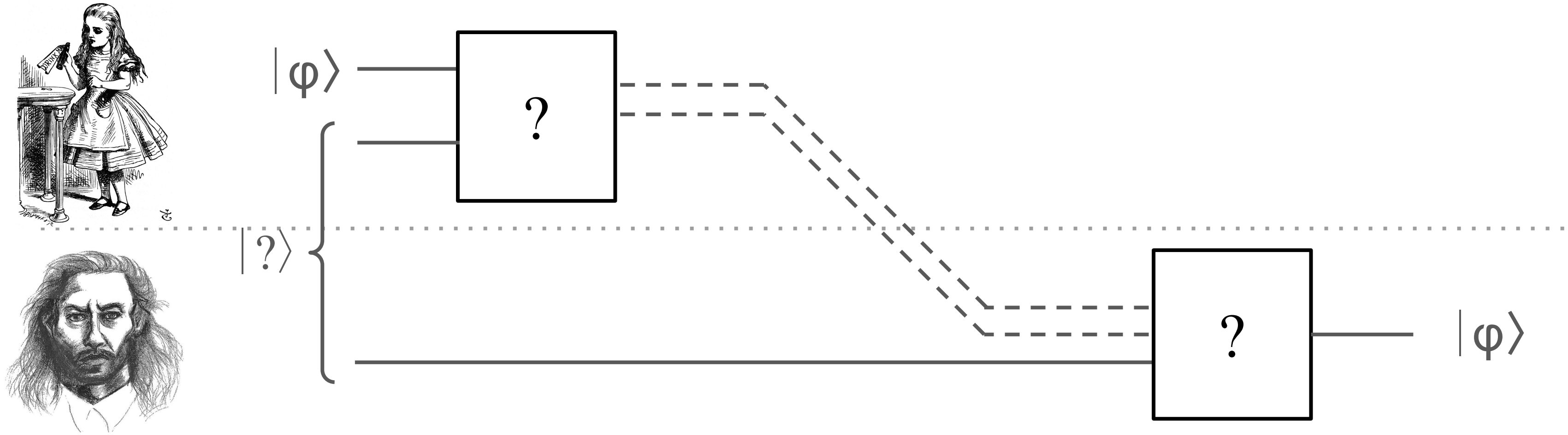
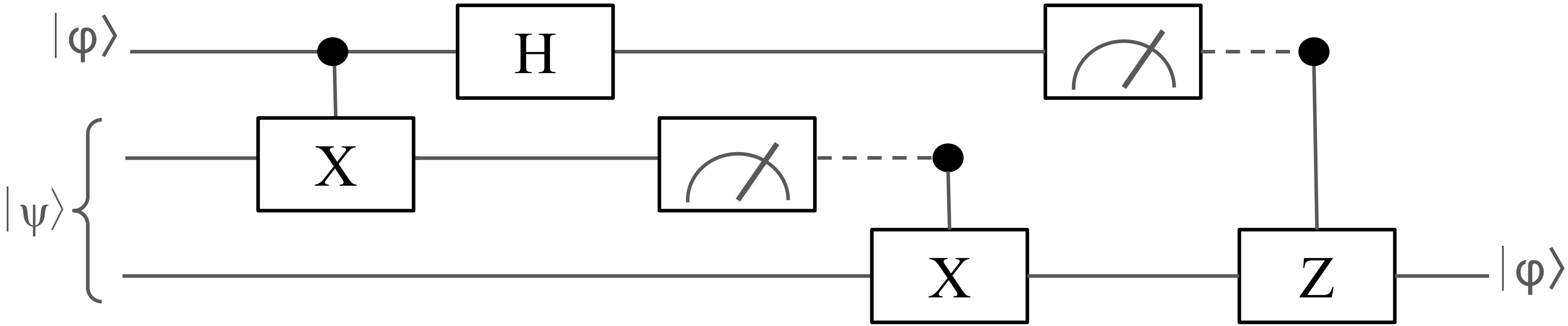


Quantum Teleportation

The solution

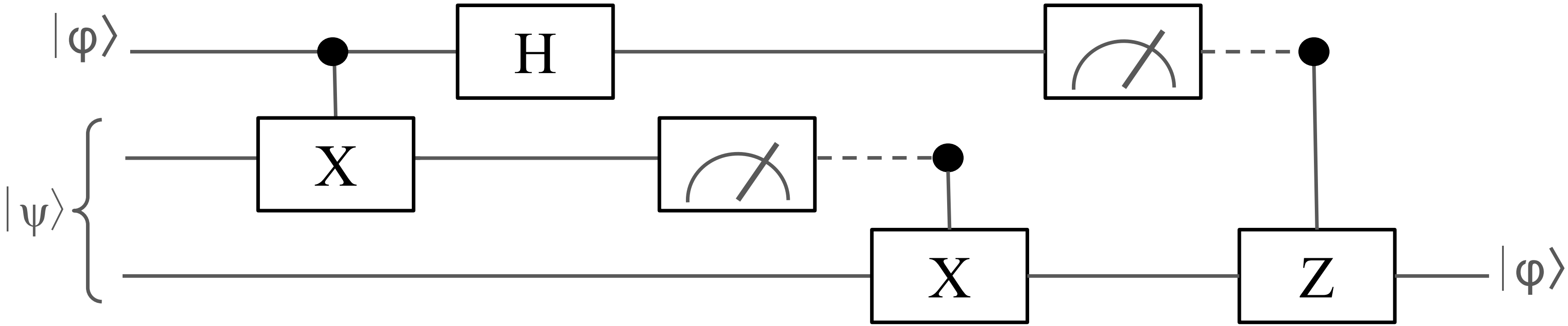
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



Quantum Teleportation - Explanation

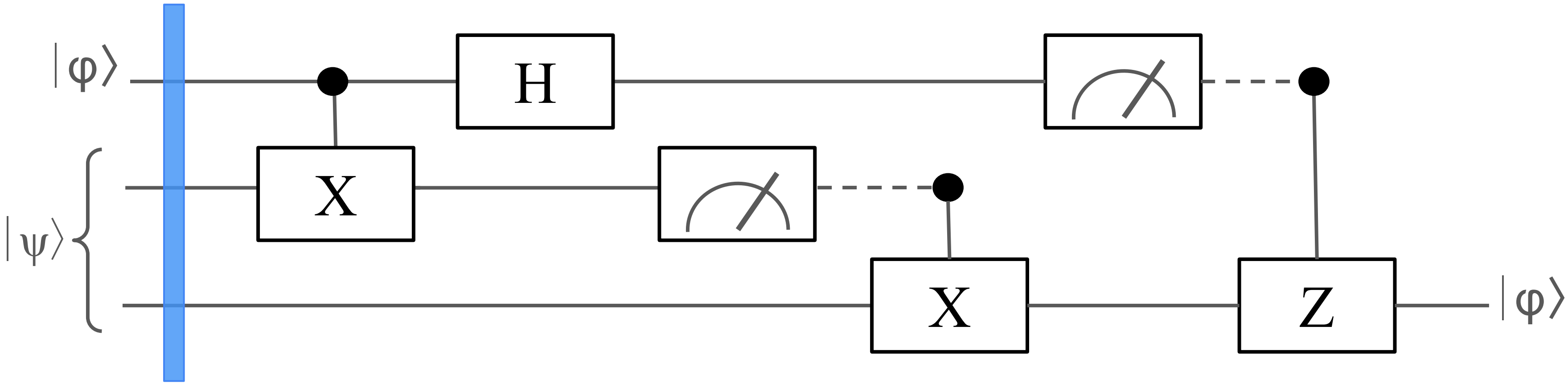
$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



Quantum Teleportation - Explanation

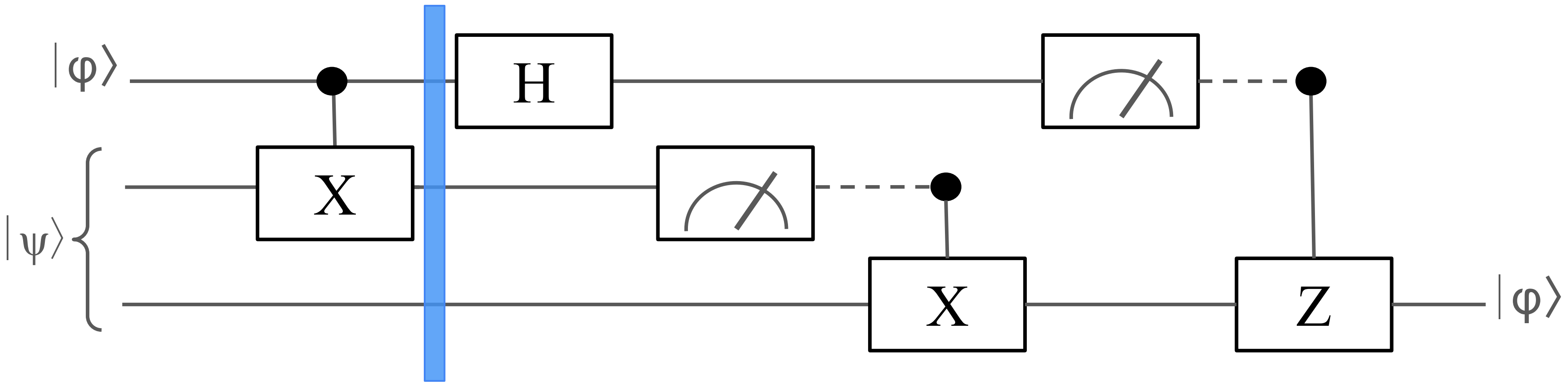
$$\frac{1}{\sqrt{2}}\left(a\,|0\rangle(\,|00\rangle + |11\rangle) + b\,|1\rangle(\,|00\rangle + |11\rangle)\right)$$



Quantum Teleportation - Explanation

$$\frac{1}{\sqrt{2}} (a\,|0\rangle\ (\ |00\rangle + |11\rangle\ ) + b\,|1\rangle\ (\ |00\rangle + |11\rangle\ ))$$

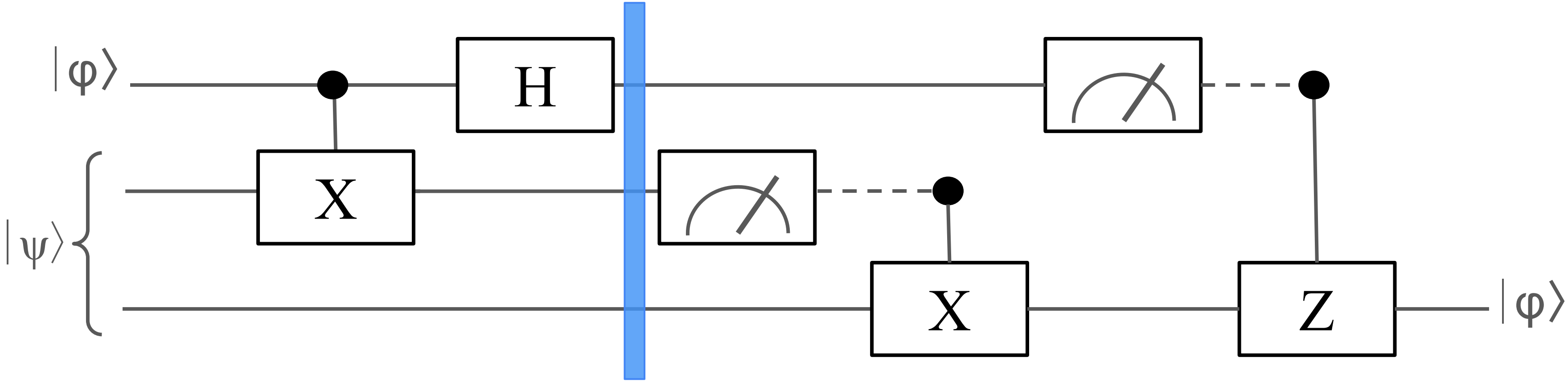
$$\frac{1}{\sqrt{2}} (a\,|0\rangle\ (\ |00\rangle + |11\rangle\ ) + b\,|1\rangle\ (\ |10\rangle + |01\rangle\ ))$$



Quantum Teleportation - Explanation

$$\frac{1}{\sqrt{2}}\left(a\boxed{|0\rangle}\left(|00\rangle+|11\rangle\right)+b\boxed{|1\rangle}\left(|10\rangle+|01\rangle\right)\right)$$

$$\frac{1}{2}\left(a\left(|0\rangle+|1\rangle\right)\left(|00\rangle+|11\rangle\right)+b\left(|0\rangle-|1\rangle\right)\left(|10\rangle+|01\rangle\right)\right)$$

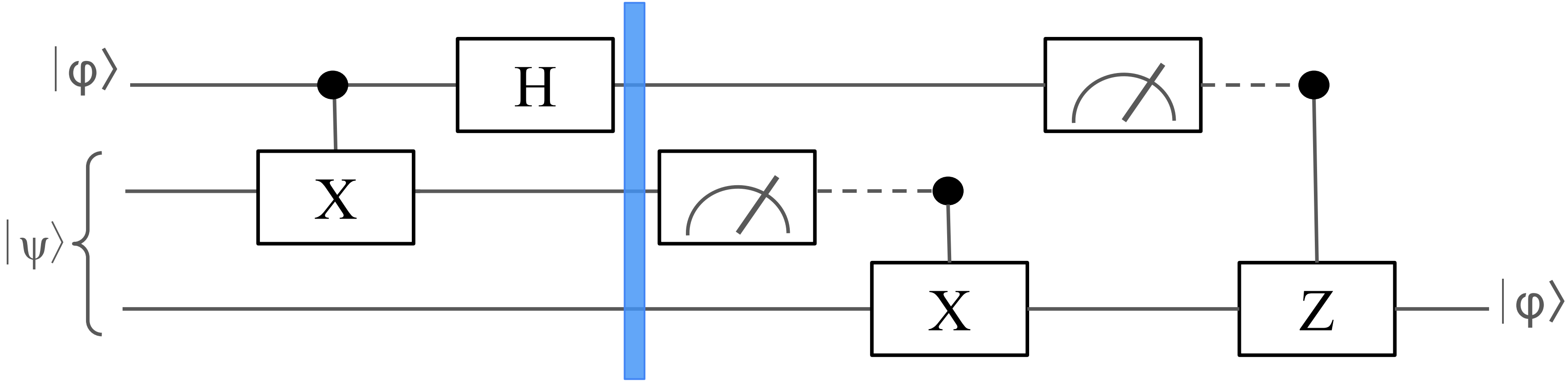


Quantum Teleportation - Explanation

$$\frac{1}{2}\,(a\,(|0\rangle + |1\rangle)\,(|00\rangle + |11\rangle) + b\,(|0\rangle - |1\rangle)(|10\rangle + |01\rangle))$$

$$\left.\frac{1}{2}\begin{array}{l}(\quad a\,(|000\rangle + |100\rangle + |011\rangle + |111\rangle) \\ \;+\, b\,(|010\rangle + |001\rangle - |110\rangle - |101\rangle))\end{array}\right\}$$

~ a single qubit



Quantum Teleportation - Explanation

$$\frac{1}{2}\Big(\, a\,(|000\rangle + |100\rangle + |011\rangle + |111\rangle)$$
$$+\, b\,(|010\rangle + |001\rangle - |110\rangle - |101\rangle)\Big)$$

Measurement of the first two qubit

$0\ \ 0 \rightarrow a|0\rangle + b|1\rangle$

$0\ \ \boxed{1} \rightarrow a|1\rangle + b|0\rangle$

$\boxed{1}\ \ 0 \rightarrow a|0\rangle - b|1\rangle$

$\boxed{1}\ \ \boxed{1} \rightarrow a|1\rangle - b|0\rangle$

X is needed

Z is needed

Models and Properties

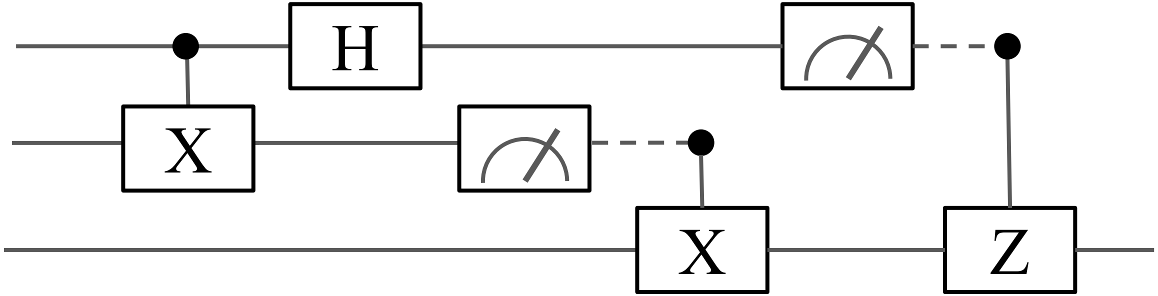
Quantum Transition System

- H - Hilbert space (**state** space for the quantum system)
- L - set of **locations** $l, l', l'', \ldots, l_0$
- l_0 - **initial** location
- T - **transitions** (l, l', U) or (l, l', M_m)

When representing Quantum Circuits

- **Transformation** gates cause **deterministic** transitions
- **Measurements** cause **nondeterministic** transitions
  - create one branch for each possible result

Quantum Teleportation as Transition System



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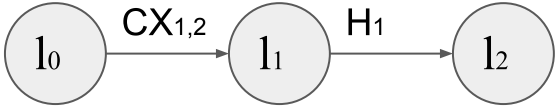

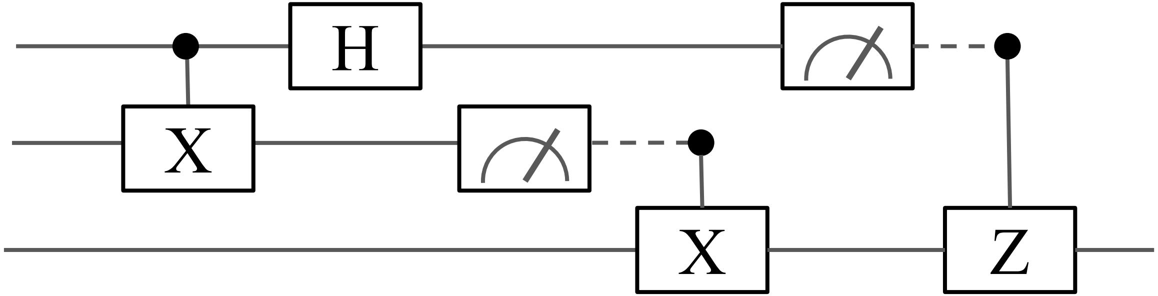

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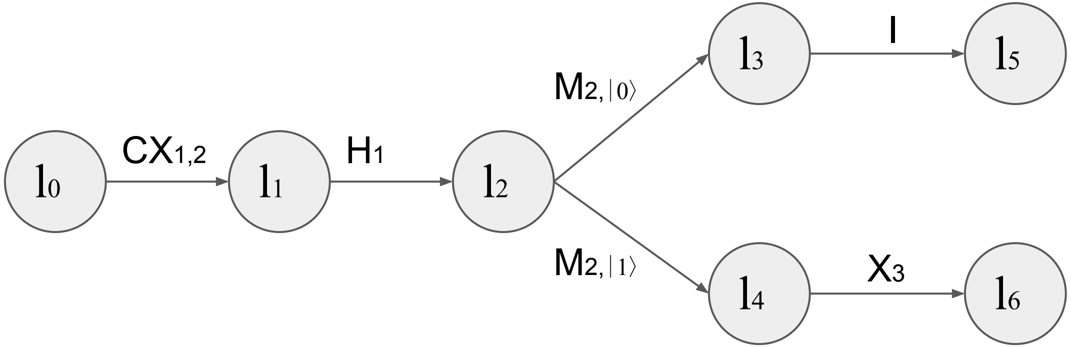

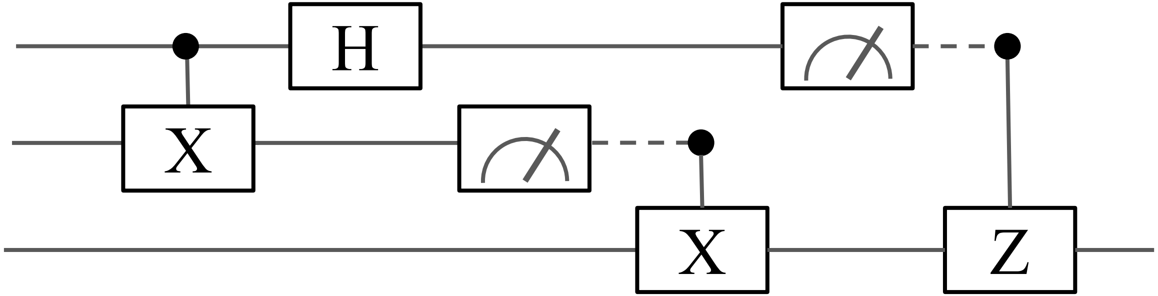

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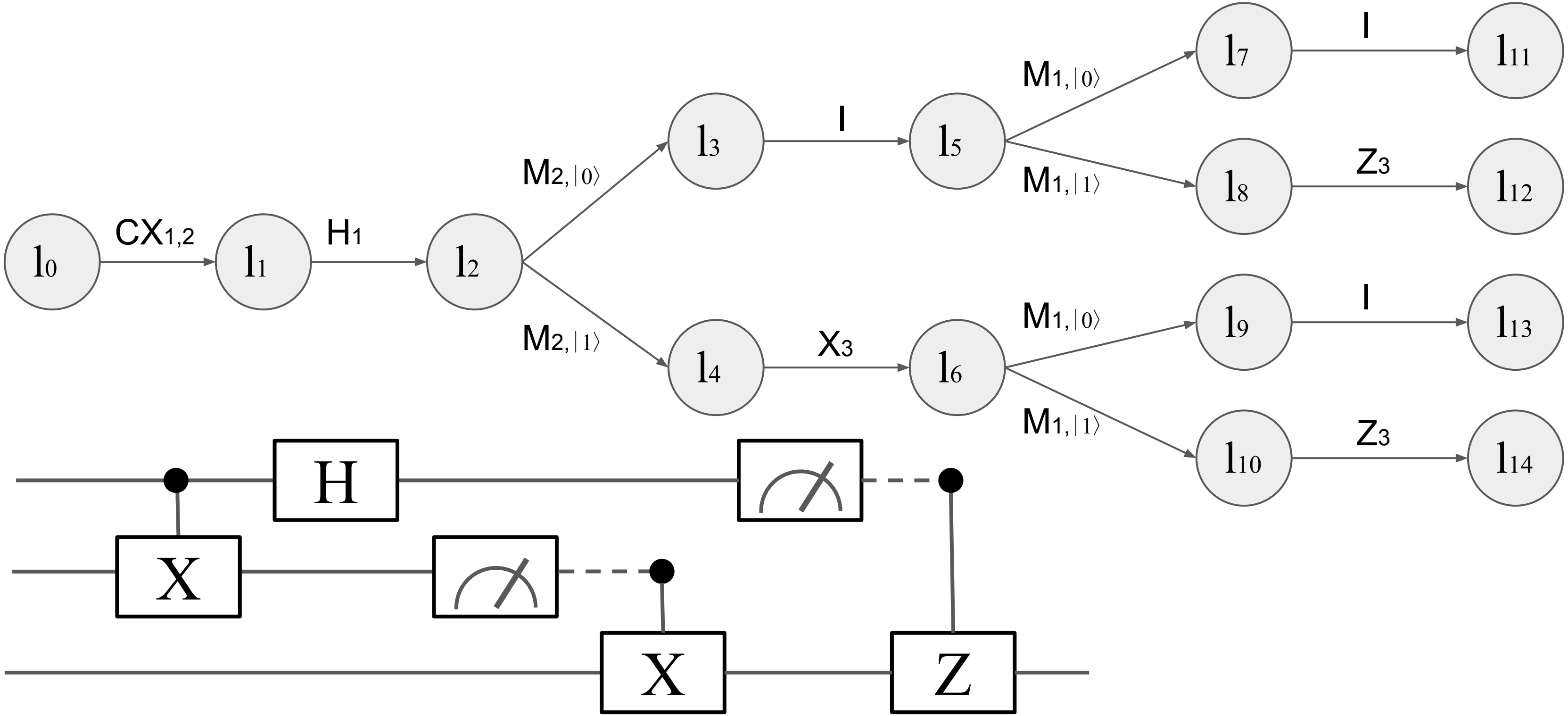


Birkhoff-von Neumann logic

H - state space of the quantum system (*Hilbert space*)

Atomic propositions χ - **closed subspaces** of H

e.g.

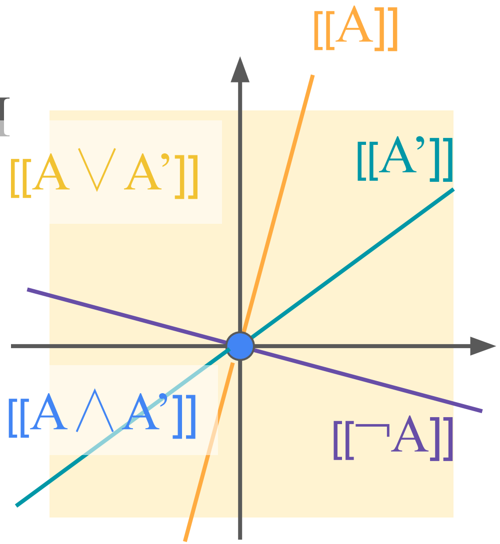
- the quantum particle has x position in the interval $[a, b]$
- the first qubit of the system is $|\varphi'\rangle$ or $-1\ |\varphi'\rangle$

$$A ::= \chi \mid \neg A \mid A \wedge A \mid A \vee A$$

Birkhoff-von Neumann logic

The semantics of a proposition A is a subset of H

$$|\varphi\rangle \vDash A \quad \text{iff} \quad |\varphi\rangle \in [[A]]$$



$$[[\chi]] = \chi$$

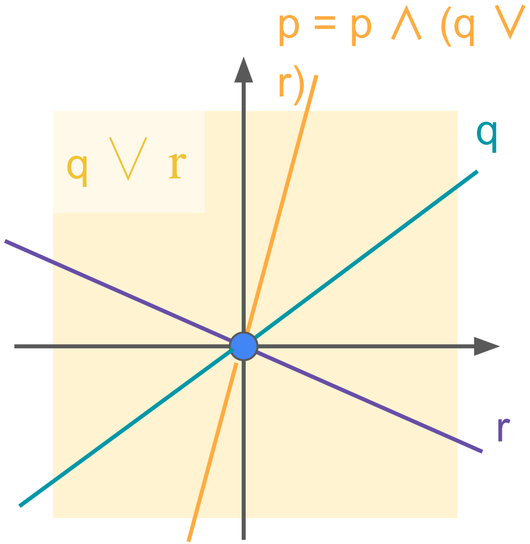
$$[[\neg A]] = \{\, |\varphi\rangle | \langle \psi | \varphi \rangle = 0,\ \psi \in [[A]] \,\}$$

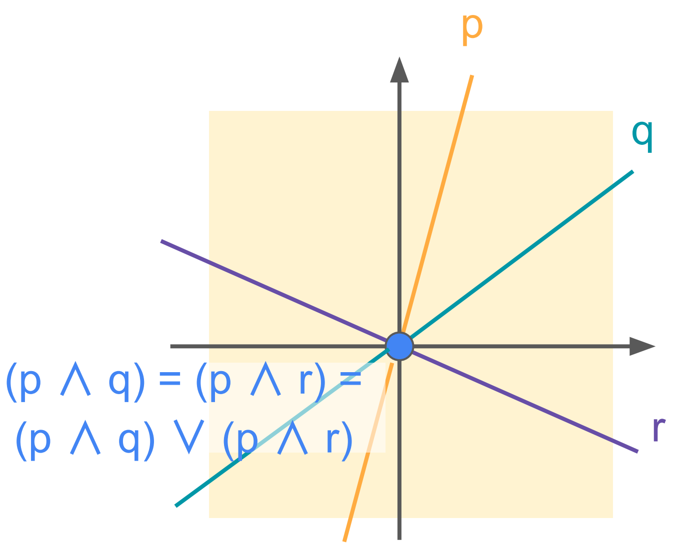
$$[[A \wedge A']] = [[A]] \cap [[A']]$$

$$[[A \vee A']] = \{\, a|\varphi\rangle + b|\psi\rangle | \ |\varphi\rangle \in [[A]],\ |\psi\rangle \in [[A']] \,\}$$

Birkhoff-von Neumann logic

$$p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$$





Temporal Extension

You can take any temporal logic with Birkhoff-von Neumann propositions instead of the classical propositions.

Computation Tree Quantum Logic

State formulas $\Phi ::= A \mid \exists\ P \mid \forall\ P \mid \neg\Phi \mid \Phi \wedge \Phi$

Path formulas $P ::= O\Phi \mid \Phi U \Phi$

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You can take any temporal logic with Birkhoff-von Neumann propositions instead of the classical propositions.

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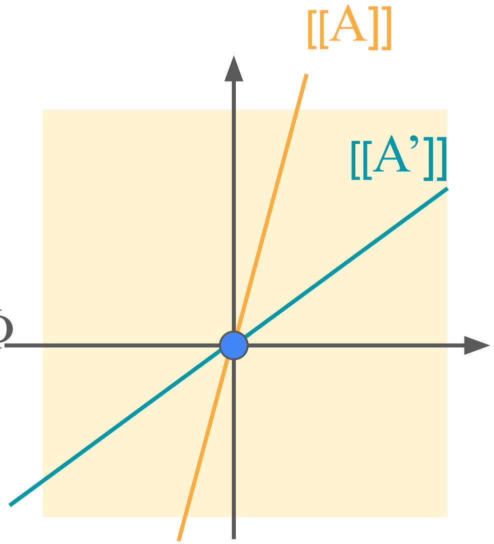

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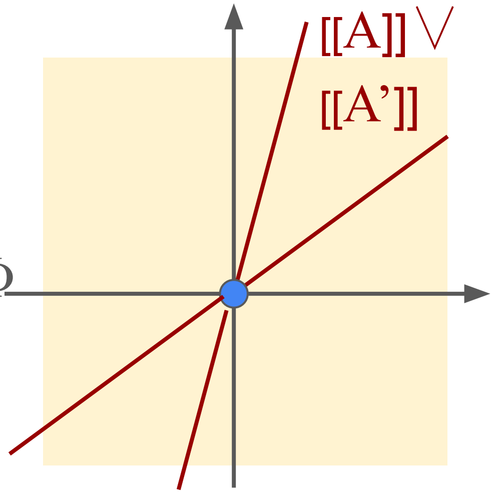


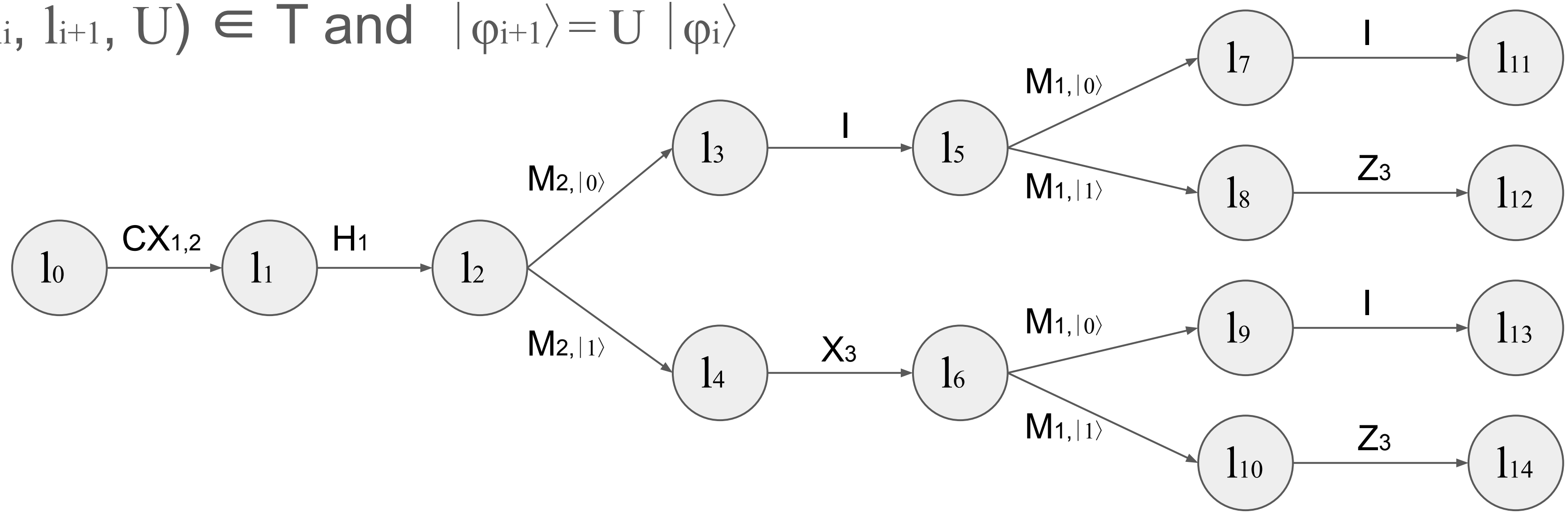
Traces of a Quantum Transition System (L, l_0, T)

Traces π are **sequences of pairs** $(l, |\varphi\rangle)$

$$(l_0, |\varphi_0\rangle)\,(l_1, |\varphi_1\rangle) \ldots (l_i, |\varphi_i\rangle) \ldots$$

s.t. for each consecutive pair $(l_i, |\varphi_i\rangle)\,(l_{i+1}, |\varphi_{i+1}\rangle)$

$(l_i, l_{i+1}, U) \in T$ and $|\varphi_{i+1}\rangle = U\,|\varphi_i\rangle$



Semantics of CTQL

$(l, |\varphi\rangle) \models A$ iff $|\varphi\rangle \in [[A]]$

$(l, |\varphi\rangle) \models \exists P$ iff $\pi \models P$ for some π starting from $(l, |\varphi\rangle)$

$(l, |\varphi\rangle) \models \forall P$ iff $\pi \models P$ for all π starting from $(l, |\varphi\rangle)$

$(l, |\varphi\rangle) \models \neg\Phi$ iff $(l, |\varphi\rangle) \not\models \Phi$

$(l, |\varphi\rangle) \models \Phi \wedge \Phi'$ iff $(l, |\varphi\rangle) \models \Phi$ and $(l, |\varphi\rangle) \models \Phi'$

$\pi \models O\Phi$ iff $\pi[1] \models O\Phi$

$\pi \models \Phi U \Phi'$ iff $\exists i.\ \pi[i] \models \Phi'$ and $\forall j < i.\ \pi[j] \models \Phi'$

Note that the satisfaction depends on the initial state $|\varphi\rangle$

Simulation-based semantics

When we check the state of the system to know if it verifies a property, **the state is not disturbed**

This means that our analysis runs on a **simulation** of the quantum circuit

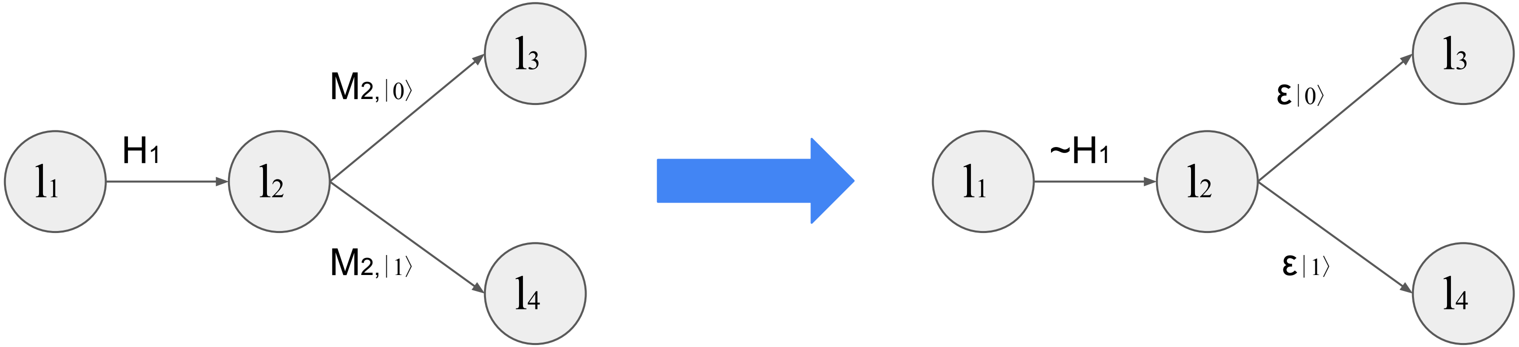
In the **measurement-based semantics**, when we check a property the system decays

Quantitative Extension of CTQL

in **Probabilistic Temporal Logic** you have $P_{[a,b]}[P]$
i.e. P is true with probability between a and b

We need to change the model

Arrows encode both transformations and (quantum) **probabilities**



Quantitative Extension of CTQL

Generalization of the classical probability measure

- **classically** we give probability $\in [0, 1]$ to an infinite path based on the probability of the finite extensions of its finite prefixes
- we can proceed **similarly** in quantum with $M \in [0, I]$

State formulas $\Phi ::= A \mid Q_{\sim M}[P] \mid \neg\Phi \mid \Phi \wedge \Phi$

Path formulas $P ::= O\Phi \mid \Phi U \Phi$

- $\sim \in \{ \sqsubseteq, \sqsupseteq, = \}$
- $M \in [0, I]$

CTQL Model Checking

CTQL Model Checking

Problem: given

- QTS $S = (H, L, l_0, T)$
- Initial state $|\varphi\rangle$
- CTQL state formula Φ

check $(S, |\varphi\rangle) \models \Phi$ [i.e. $(l_0, |\varphi\rangle) \models \Phi$]

We build a **classical** Transition System $S'_{|\varphi\rangle}$ s.t.

$(S, |\varphi\rangle) \models \Phi$ iff $S'_{|\varphi\rangle} \models \Phi$

CTQL reduced to CTL

$S'_{|\varphi\rangle} = (L', (l_0, |\varphi\rangle), T', Ap, Lab)$

where:

- $L' = L \times H$
- $((l_i, |\varphi_i\rangle), (l_j, |\varphi_j\rangle)) \in T'$ iff $(l_i, l_j, U) \in T$ and $|\varphi_j\rangle = U|\varphi_i\rangle$
- Ap is the set of Birkhoff-von Neumann propositions
- $A \in Lab(l_i, |\varphi_i\rangle)$ iff $|\varphi_i\rangle \vDash A$

Theorem

$(S, |\varphi\rangle) \vDash \Phi$ iff $S'_{|\varphi\rangle} \vDash \Phi$

CTQL reduced to CTL

$S'_{|\varphi\rangle} = (L', (l_0, |\varphi\rangle), T', Ap, Lab)$

where:

- $L' = L \times H$ (Actually just the reachable configurations)
- $((l_i, |\varphi_i\rangle), (l_j, |\varphi_j\rangle)) \in T'$ iff $(l_i, l_j, U) \in T$ and $|\varphi_j\rangle = U|\varphi_i\rangle$
- Ap is the set of Birkhoff-von Neumann propositions
- $A \in Lab(l_i, |\varphi_i\rangle)$ iff $|\varphi_i\rangle \models A$

Theorem

$(S, |\varphi\rangle) \models \Phi$ iff $S'_{|\varphi\rangle} \models \Phi$

Reachability Analysis of Quantum Circuits

A simpler case:

the system evolves as described by ε (Quantum Markov Chain)

The image of a subspace X under ε is

$$\varepsilon(X) = \mathrm{span}\left(\bigcup_{|\varphi\rangle \in X} \mathrm{supp}(\varepsilon(|\varphi\rangle\langle\varphi|))\right)$$

We actually need the states reachable with $\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3 \ldots$

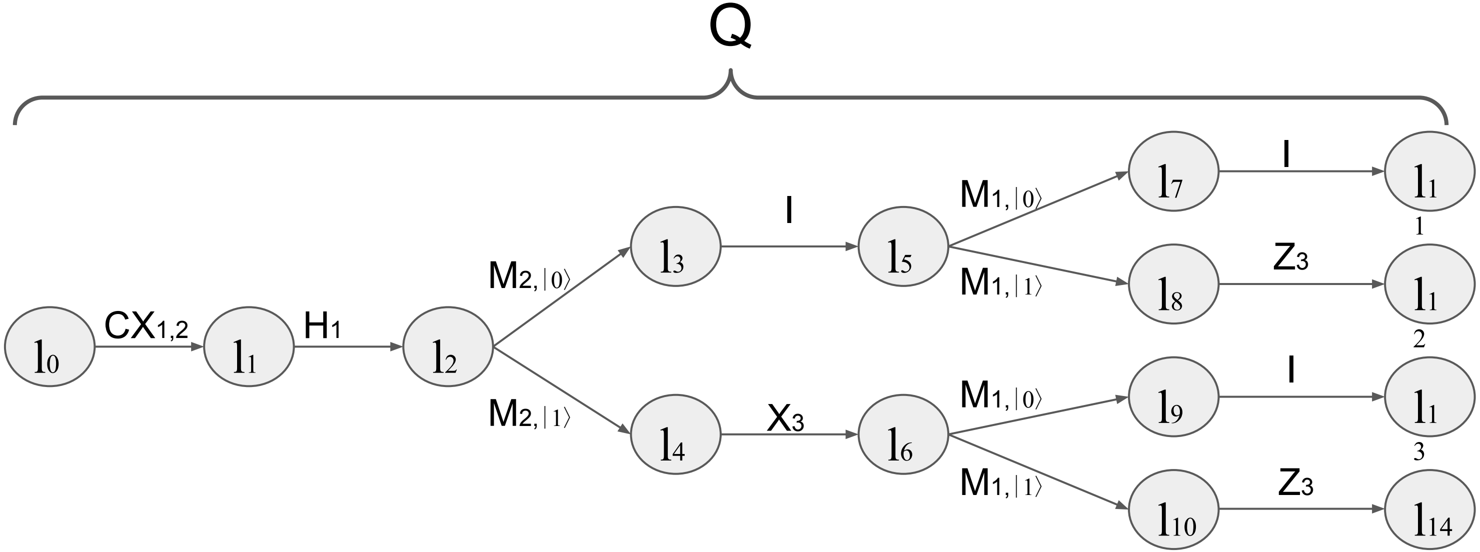
Theorem

$$\mathrm{span}\left(\bigcup_{i=0\ldots d} \mathrm{supp}(\varepsilon^i(\{|\varphi\rangle\}))\right) = \mathrm{supp}\left(\sum_{i=0\ldots d} \varepsilon^i(\{|\varphi\rangle\})\right)$$

Reachability Analysis of Quantum Circuits

L’ = configurations reachable from (l_0, $|\varphi\rangle$)

Compute the reachable subspace w.r.t. Q ∋ (l, l’, ε)



Something More

Dealing with Mixed States

- **Pure states** (superposition of classic states)

  $|\varphi\rangle = a|0\rangle + b|1\rangle$

  $a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1$
- **Mixed states** are **classical mixture** of pure quantum states

  $\{ (|\varphi_i\rangle, \ p_i) \}$ s.t. $\forall$i. $p_i \geq 1$ and $\sum_i \ p_i = 1$
  - The system is in state $|\varphi_i\rangle$ with probability p_i

Represent missing information and not isolated states
e.g. one qubit of an entangled pair

Dynamics of Mixed States

- Mixed states are represented by **density matrices** ρ

  $\{ (|\varphi_i\rangle, \ p_i) \}$

  $\rho = \sum_i \ p_i \ |\varphi_i\rangle\langle\varphi_i|$
- Isolated System Evolution $\rho \rightarrow \rho'$
  - Unitary transformation $\rho' = U \rho U^\dagger$
  - Measurement $\rho' = M_m \rho M_m^\dagger / tr (M_m^\dagger M_m \rho)$
- Open System Evolution $\rho' = \varepsilon (\rho)$
  - ε is a Linear Transformation (super-operator) s.t. …

Super-operators

- Can represent Unitary Operators U

  $$\varepsilon(\rho) = U \rho U^\dagger$$

- Can represent the decay for a measurement with result m

  $$\varepsilon_m(\rho) = M_m \rho M_m^\dagger$$

- Can represent the decay for a measurement

  $$\varepsilon(\rho) = \sum_m M_m \rho M_m^\dagger$$

- **Can represent quantum noises and noisy gates**

Open Systems

- Given a composite system $S + E$ in state $\rho_{S+E} \in H_{S+E}$
- The state of the **subsystem** S is defined as

$$\rho_S = tr_E (\rho_{S+E})$$

- And its evolution is according a super-operator ε

Open Systems

Closed Composite System

Open Subsystem

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\rho_{S+E}(t_0) \arrow[r, shift left, "\varepsilon_U"] \arrow[d, "\mathrm{tr}_E"'] & \rho_{S+E}(t_1) \arrow[l, shift left, "\varepsilon_U^{-1}"] \arrow[d, "\mathrm{tr}_E"'] \\
\rho_S(t_0) \arrow[r, "\varepsilon"] & \rho_S(t_1)
\end{tikzcd}
$$

Crossing the line causes a lost of information:you cannot go back

Model Checking with Mixed States

- Everything seen so far **works with mixed states** ρ
  - In quantum transition systems:
    arrows are labeled with super-operators ε
  - In Quantum Logic: $\rho \models A \quad \text{iff} \quad \text{supp}(\rho) \subseteq [[A]]$
  - In CTQL: $(l, \rho) \models A \quad \text{iff} \quad \text{supp}(\rho) \subseteq [[A]]$

- **We can model check noisy circuits!**

Optimization via Tensor Networks

A Tensor is a **multidimensional matrix with named indexes**.

Formally:

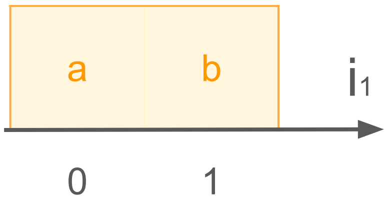
Given a set of indexes $\bar{I} = (i_1, \ldots i_n)$,

a Tensor is a mapping

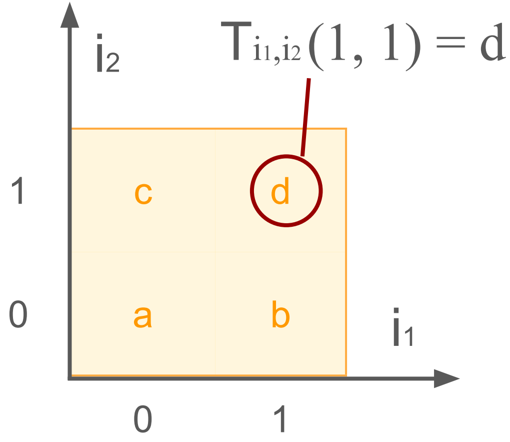
$$T : \{0, 1\}^{\bar{I}} \longrightarrow \mathbb{C}$$

Tensor Representation of Quantum States

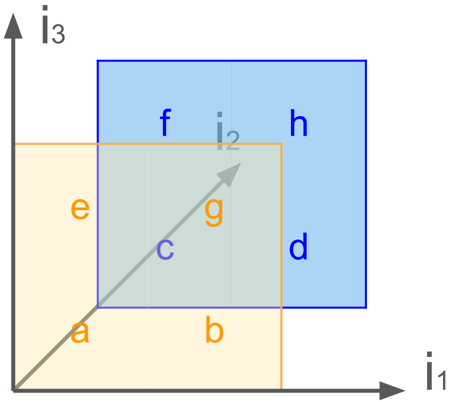
- Single qubit
  $|\varphi\rangle = a|0\rangle + b|1\rangle$



- Pair of qubits
  $|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$



- Triplet of qubits
  $|\varphi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle$
  $+ \ e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle$



Tensor Representation of Gates

Gates on n qubits can be represented as tensors with indices $(\underbrace{i_1, \ldots i_n}_{\text{inputs}}, \underbrace{i_1', \ldots i_n'}_{\text{outputs}})$

Tensor Network is a hyper-graph with

- Tensors as nodes
- hyper-edges are the **shared indexes**

Tensor Contraction

A generalization of Matrix product

- T_1 on indices $\bar{I}_1\bar{I}_c$
- T_2 on indices $\bar{I}_2\bar{I}_c$

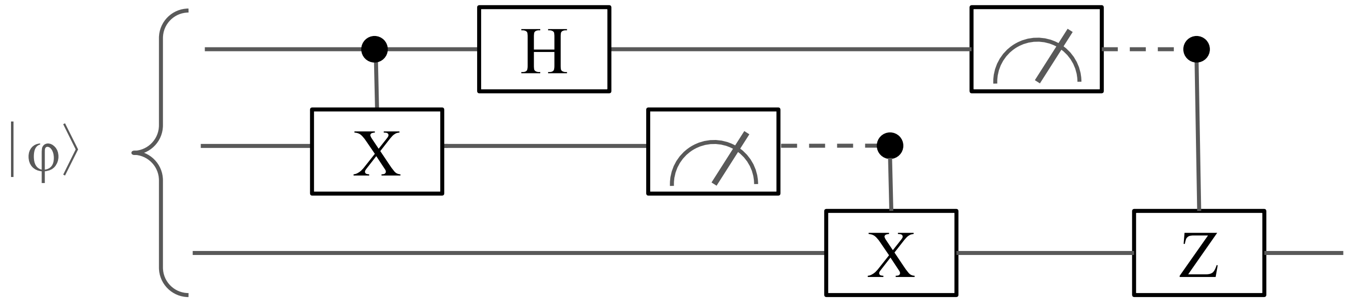
Contraction returns a Tensor T' on indices $\bar{I}_1\bar{I}_2$

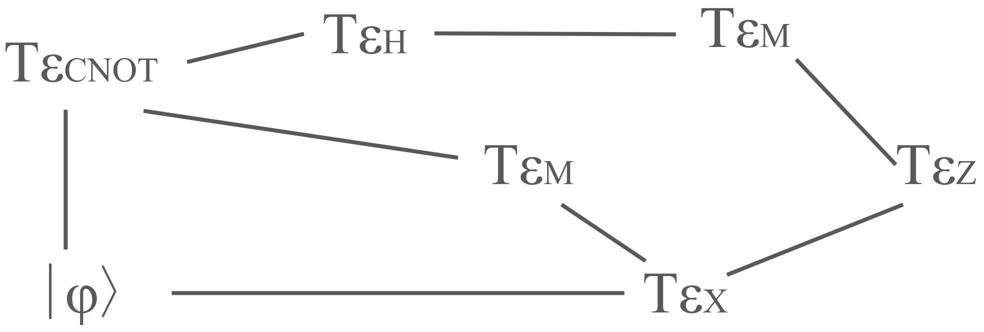
$$T'_{\bar{I}_1\bar{I}_2}(\bar{a},\bar{e}) = \sum_{\bar{o}\in\{0,1\}^{\bar{I}_c}} T_{\bar{I}_1\bar{I}_c}(\bar{a},\bar{o}) \cdot T_{\bar{I}_2\bar{I}_c}(\bar{e},\bar{o})$$

- **Composing** transformations
- **Applying** transformation to qubits

In any order

Tensor Contraction





Why Tensor Networks

- Contraction cost depends on the **actual information** stored in the system (linked to entanglement)
- You can choose **any order**
- Thus you can **exploit regularity and locality** in the quantum circuit

Conclusions

Conclusions

- We have seen
  - Quantum Transition Systems
  - CTQL
  - Reduction to CTL model checking
  - Works with Open Systems and noisy gates
  - Optimization via Tensor Networks
- With a small comparison w.r.t.
  - More expressive modeling and logics

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