Model Checking Quantum Circuits
Approach

- Based on the paper by Ying
- A pragmatic approach
- We will give some context and reason about choices

Model Checking for Verification of Quantum Circuits

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Abstract. In this talk, we will describe a framework for assertion-based verification (ABV) of quantum circuits by applying model checking techniques for quantum systems developed in our previous work, in which:
- Noiseless and noisy quantum circuits are modelled as operator- and superoperator-valued matrices in the study.
Content

- Brief Introduction to Quantum Circuits
- **Modeling** Quantum Circuits as Transition Systems
- **A Logic** for Temporal Properties on Quantum Systems
- Reduction to CTL **model checking**
- Dealing with **Mixed States**
- **Optimization** via Tensor Networks
Quantum Circuits
Qubits

- Two **classic states** $|0\rangle$ and $|1\rangle$
- Pure states (**superposition** of classic states)
  $$|\varphi\rangle = a|0\rangle + b|1\rangle$$
  $$a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1$$

- **probability** of each classic state
- **wave** phase (interference)

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Measurements

**Outcome** on $|\varphi\rangle = a |0\rangle + b |1\rangle$

- $|0\rangle$ with probability $|a|^2$
- $|1\rangle$ with probability $|b|^2$

The system **decays** in the observed classical state

- $|\varphi'\rangle = 1 |0\rangle + 0 |1\rangle$
- $|\varphi''\rangle = 0 |0\rangle + 1 |1\rangle$
Dynamics of an (isolated) quantum system

\[ |\varphi\rangle \rightarrow |\varphi'\rangle \text{ with } |\varphi'\rangle = U |\varphi\rangle \]

$U$ is a transformation
- **Linear:** $U (a |0\rangle + b |1\rangle) = a U |0\rangle + b U |1\rangle$
- **Unitary:** $U^\dagger U = U U^\dagger = I$
Single Qubit Transformations (Gates)

\[ |\varphi\rangle \rightarrow |\varphi'\rangle \ \text{with} \ |\varphi'\rangle = U |\varphi\rangle \]
where \[ |\varphi\rangle = a |0\rangle + b |1\rangle \]

\[ U |\varphi\rangle = a |0\rangle + b |1\rangle \]
\[ = |0\rangle + \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]
\[ = |1\rangle - \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

\begin{align*}
|0\rangle & \quad \text{X gate} \quad |1\rangle & \quad \text{Z gate} \\
|0\rangle & \quad \text{H gate} \quad |1\rangle & \quad \text{H gate}
\end{align*}
Multiple Qubits Systems

- **Classic states** $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$
- **Pure states**
  
  $|\varphi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

  \[a, b, c, d \in \mathbb{C} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\]
Entangled States

\[ |\varphi\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \]

- \( |0\rangle \) or \( |1\rangle \) with equal probability for both qubits
- BUT they **must be equal**
  - when one is measured, both of them decay

You **cannot decompose** the system in two components
Two Qubits Transformations (Gates)

\[ |\varphi\rangle \rightarrow |\varphi'\rangle \text{ with } |\varphi'\rangle = U |\varphi\rangle \]

where \[ |\varphi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \]

Only a specific case

- **control** qubit

- **target** qubit

\[ |0,x\rangle \rightarrow |0,x\rangle \]
\[ |1,x\rangle \rightarrow |1,Ux\rangle \]
Controlled NOT

$|\varphi\rangle \rightarrow |\varphi'\rangle$

**CNOT** can create entanglement

$|\varphi\rangle$ is non entangled

\[
\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)
\]

$|\varphi'\rangle$ is entangled

\[
\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]
Measurements (Again)

Outcome of measuring the **first** qubit of
\[ |\varphi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \]

- \(|0\rangle\) with probability \(p(0) = |a|^2 + |b|^2\)
- \(|1\rangle\) with probability \(p(1) = |c|^2 + |d|^2\)

The system **decays** according to the observed classical state

**Operators** that applied to \(|\varphi\rangle\) returns the new state:

- \(M_{|0\rangle} / \sqrt{p(0)}\) where \(M_{|0\rangle} = |0\rangle\langle0|\) with \(\langle0| = (1\ 0)\)
- \(M_{|1\rangle} / \sqrt{p(1)}\) where \(M_{|1\rangle} = |1\rangle\langle1|\) with \(\langle1| = (0\ 1)\)
Combinatorial Quantum Circuits

Just a composition of gates on qubit wires
Measurements only at the end of computation
Dynamic Quantum Circuits

- **Quantum** bit wires
- **Classical** bit wires
- Quantum **Gates** (also controlled by classical bits)
- **Measurements** in arbitrary points of the computation
Quantum Teleportation

- Alice sends $\lvert \varphi \rangle$ to Bob using classical communication.
- They can start with entangled qubits.
Quantum Teleportation

- Alice send $|\phi\rangle$ to Bob using classical communication
- They can start with entangled qubits
Quantum Teleportation

The solution

\[ |\psi\rangle = \frac{1}{\sqrt{2}} ( |00\rangle + |11\rangle ) \]
Quantum Teleportation - Explanation

\[ |\phi\rangle = a |0\rangle + b |1\rangle \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]
Quantum Teleportation - Explanation

$$\frac{1}{\sqrt{2}} (a \left| 0 \right\rangle (\left| 00 \right\rangle + \left| 11 \right\rangle) + b \left| 1 \right\rangle (\left| 00 \right\rangle + \left| 11 \right\rangle))$$
Quantum Teleportation - Explanation

\[ \frac{1}{\sqrt{2}} \left( a \left( |0\rangle + |1\rangle \right) + b |1\rangle \left( |0\rangle + |1\rangle \right) \right) \]

\[ \frac{1}{\sqrt{2}} \left( a |0\rangle + |1\rangle \right) + b |1\rangle \left( |10\rangle + |01\rangle \right) \]
Quantum Teleportation - Explanation

\[
\frac{1}{\sqrt{2}} \left( a |0\rangle ( |00\rangle + |11\rangle ) + b |1\rangle ( |10\rangle + |01\rangle ) \right)
\]

\[
\frac{1}{2} \left( a ( |0\rangle + |1\rangle ) ( |00\rangle + |11\rangle ) + b ( |0\rangle - |1\rangle ) ( |10\rangle + |01\rangle ) \right)
\]
Quantum Teleportation - Explanation

\[ \frac{1}{2} (a (|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b (|0\rangle - |1\rangle)(|10\rangle + |01\rangle)) \]

\[ \frac{1}{2} (a (|000\rangle + |100\rangle + |011\rangle + |111\rangle) + b (|010\rangle + |001\rangle - |110\rangle - |101\rangle)) \] ~ a single qubit
Quantum Teleportation - Explanation

$$\frac{1}{2} \left( a \left( |000\rangle + |100\rangle + |011\rangle + |111\rangle \right) + b \left( |010\rangle + |001\rangle - |110\rangle - |101\rangle \right) \right)$$

Measurement of the first two qubits:

0  0 → a |0⟩ + b |1⟩
0  1 → a |1⟩ + b |0⟩
1  0 → a |0⟩ - b |1⟩
1  1 → a |1⟩ - b |0⟩

X is needed
Z is needed
Models and Properties
Quantum Transition System

- **H** - Hilbert space (state space for the quantum system)
- **L** - set of locations \( l, l', l'', \ldots, l_0 \)
- **\( l_0 \)** - initial location
- **T** - transitions \((l, l', U)\) or \((l, l', M_m)\)

When representing Quantum Circuits
- **Transformation** gates cause deterministic transitions
- **Measurements** cause nondeterministic transitions
  - create one branch for each possible result
Quantum Teleportation as Transition System
Quantum Teleportation as Transition System
Quantum Teleportation as Transition System
Quantum Teleportation as Transition System
Birkhoff-von Neumann logic

\( \mathbb{H} \) - state space of the quantum system (Hilbert space)

Atomic propositions \( \chi \) - closed subspaces of \( \mathbb{H} \)

- the quantum particle has position in the interval \([a, b]\)
- the first qubit of the system is \(|\varphi'\rangle\) or \(-1\,|\varphi'\rangle\)

\[
A ::= \chi | \neg A | A \land A | A \lor A
\]
Birkhoff-von Neumann logic

The semantics of a proposition $A$ is a subset of $H$

$|\varphi\rangle \models A$ iff $|\varphi\rangle \in [[A]]$

$[[\chi]] = \chi$

$[[\neg A]] = \{ |\varphi\rangle \langle \psi | \varphi \rangle = 0, \psi \in [[A]] \}$

$[[A \land A']] = [[A]] \cap [[A']]$

$[[A \lor A']] = \{ a|\varphi\rangle + b|\psi\rangle | |\varphi\rangle \in [[A]], |\psi\rangle \in [[A']] \}$
Birkhoff-von Neumann logic

\[ p \land (q \lor r) \neq (p \land q) \lor (p \land r) \]

\[ p = p \land (q \lor r) \]

\[ q \lor r \]

\[ (p \land q) = (p \land r) = (p \land q) \lor (p \land r) \]
Temporal Extension

You can take any temporal logic with Birkhoff-von Neumann propositions instead of the classical propositions.

Computation Tree Quantum Logic

State formulas \( \Phi ::= A | \exists P | \forall P | \neg \Phi | \Phi \land \Phi \)

Path formulas \( P ::= O \Phi | \Phi U \Phi \)
Temporal Extension

You can take any temporal logic with Birkhoff-von Neumann propositions instead of the classical propositions.

Computation Tree Quantum Logic

State formulas  $\Phi ::= A \mid \exists P \mid \forall P \mid \neg \Phi \mid \Phi \land \Phi$

Path formulas  $P ::= O\Phi \mid \Phi U \Phi$
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Computation Tree Quantum Logic

State formulas $\Phi ::= A \mid \exists P \mid \forall P \mid \neg \Phi \mid \Phi \land \Phi$

Path formulas $P ::= O\Phi \mid \Phi U \Phi$
Traces of a Quantum Transition System \((L, l_0, T)\)

Traces \(\pi\) are sequences of pairs \((l, |\varphi\rangle)\)

\[
(l_0, |\varphi_0\rangle) (l_1, |\varphi_1\rangle) \ldots (l_i, |\varphi_i\rangle) \ldots
\]

s.t. for each consecutive pair \((l_i, |\varphi_i\rangle) (l_{i+1}, |\varphi_{i+1}\rangle)\)

\((l_i, l_{i+1}, U) \in T\) and \(|\varphi_{i+1}\rangle = U |\varphi_i\rangle\)

\[
\begin{array}{ccccc}
l_0 & \xrightarrow{\text{CX}_{1,2}} & l_1 & \xrightarrow{H_1} & l_2 \\
& & & \xrightarrow{\text{M}_2,|0\rangle} & l_3 \\
& & & & \xrightarrow{l} l_5 \\
& & & \xrightarrow{\text{M}_2,|1\rangle} & l_4 \\
& & & \xrightarrow{X_3} & l_6 \\
& & & & \xrightarrow{\text{I}_1} l_7 \\
& & & & \xrightarrow{\text{M}_1,|0\rangle} l_8 \\
& & & & \xrightarrow{\text{M}_1,|1\rangle} l_9 \\
& & & & \xrightarrow{\text{M}_1,|0\rangle} l_{10} \\
& & & & \xrightarrow{\text{M}_1,|1\rangle} l_{11} \\
& & & & \xrightarrow{Z_3} l_{12} \\
& & & & \xrightarrow{Z_3} l_{13} \\
& & & & \xrightarrow{\text{I}_1} l_{14} \\
\end{array}
\]
Semantics of CTQL

\[(1, |\varphi\rangle) \models A \iff |\varphi\rangle \in [[A]]\]
\[(1, |\varphi\rangle) \models \exists P \iff \pi \models P \text{ for some } \pi \text{ starting from } (1, |\varphi\rangle)\]
\[(1, |\varphi\rangle) \models \forall P \iff \pi \models P \text{ for all } \pi \text{ starting from } (1, |\varphi\rangle)\]
\[(1, |\varphi\rangle) \models \neg \Phi \iff (1, |\varphi\rangle) \not\models \Phi\]
\[(1, |\varphi\rangle) \models \Phi \land \Phi' \iff (1, |\varphi\rangle) \models \Phi \text{ and } (1, |\varphi\rangle) \models \Phi'\]
\[\pi \models O\Phi \iff \pi[1] \models O\Phi\]
\[\pi \models \Phi U \Phi' \iff \exists i. \pi[i] \models \Phi' \text{ and } \forall j < i. \pi[j] \models \Phi'\]

Note that the satisfaction depends on the initial state \[|\varphi\rangle\]
Simulation-based semantics

When we check the state of the system to know if it verifies a property, the state is not disturbed.

This means that our analysis runs on a simulation of the quantum circuit.

In the measurement-based semantics, when we check a property the system decays.
Quantitative Extension of CTQL

in Probabilistic Temporal Logic you have $P_{[a,b]}[P]$ i.e. $P$ is true with probability between $a$ and $b$

We need to change the model

Arrows encode both transformations and (quantum) *probabilities*
Quantitative Extension of CTQL

Generalization of the classical probability measure
- **classically** we give probability $\in [0, 1]$ to an infinite path based on the probability of the finite extensions of its finite prefixes
- we can proceed *similarly* in quantum with $M \in [0, I]$

State formulas \[ \Phi ::= A \mid Q^{-M}[P] \mid \neg \Phi \mid \Phi \land \Phi \]
Path formulas \[ P ::= O\Phi \mid \Phi U \Phi \]

- $\sim \in \{ \subseteq, \supseteq, = \}$
- $M \in [0, I]$
CTQL Model Checking
CTQL Model Checking

**Problem:** given

- QTS $S = (H, L, l_0, T)$
- Initial state $|\varphi\rangle$
- CTQL state formula $\Phi$

check $(S, |\varphi\rangle) \models \Phi$ \hspace{1cm} [ i.e. $(l_0, |\varphi\rangle) \models \Phi$ ]

We build a **classical** Transition System $S'|\varphi\rangle$s.t.

$(S, |\varphi\rangle) \models \Phi$ \hspace{0.5cm} iff \hspace{0.5cm} $S'|\varphi\rangle \models \Phi$
CTQL reduced to CTL

\[ S' \mid \varphi \rangle = (L', (l_0, \mid \varphi \rangle), T', Ap, Lab) \]

where:

- \[ L' = L \times H \]
- \[ ((l_i, \mid \varphi_i \rangle), (l_j, \mid \varphi_j \rangle)) \in T' \text{ iff } (l_i, l_j, U) \in T \text{ and } \mid \varphi_j \rangle = U \mid \varphi_i \rangle \]
- \( Ap \) is the set of Birkhoff-von Neumann propositions
- \( A \in Lab (l_i, \mid \varphi_i \rangle) \text{ iff } \mid \varphi_i \rangle \models A \)

**Theorem**

\[ (S, \mid \varphi \rangle) \models \Phi \text{ iff } S' \mid \varphi \rangle \models \Phi \]
CTQL reduced to CTL

\[ S' |\varphi\rangle = (L', ( l_0, |\varphi\rangle), T', Ap, Lab) \]

where:

- \( L' = L \times H \) (Actually just the reachable configurations)
- \( (( l_i, |\varphi_i\rangle), ( l_j, |\varphi_j\rangle)) \in T' \) iff \( (l_i, l_j, U) \in T \) and \( |\varphi_j\rangle = U |\varphi_i\rangle \)
- \( Ap \) is the set of Birkhoff-von Neumann propositions
- \( A \in Lab ( l_i, |\varphi_i\rangle) \) iff \( |\varphi_i\rangle \vDash A \)

Theorem

\( ( S, |\varphi\rangle) \vDash \Phi \) iff \( S' |\varphi\rangle \vDash \Phi \)
Reachability Analysis of Quantum Circuits

A simpler case:
the system evolves as described by $\varepsilon$ (Quantum Markov Chain)

The image of a subspace $X$ under $\varepsilon$ is

$$\varepsilon(X) = \text{span} \left( \bigcup_{|\phi\rangle \in X} \text{supp}(\varepsilon(|\phi\rangle\langle\phi|)) \right)$$

We actually need the states reachable with $\varepsilon^0$, $\varepsilon^1$, $\varepsilon^2$, $\varepsilon^3$...

**Theorem**

$$\text{span} \left( \bigcup_{i=0}^{d} \text{supp}(\varepsilon(|\phi\rangle)\{i\}) \right) = \text{supp} \left( \sum_{i=0}^{d} \varepsilon(|\phi\rangle)\{i\} \right)$$
Reachability Analysis of Quantum Circuits

$L' = \text{configurations reachable from (} l_0, |\phi\rangle \text{)}$

Compute the reachable subspace w.r.t. $Q \ni (l, l', \epsilon)$
Something More
Dealing with Mixed States

- **Pure states** (superposition of classic states)
  \[ |\varphi\rangle = a |0\rangle + b |1\rangle \]
  \[ a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1 \]

- **Mixed states** are **classical mixture** of pure quantum states
  \[ \{ (|\varphi_i\rangle, p_i) \} \quad \text{s.t. } \forall i. \quad p_i \geq 1 \text{ and } \sum_i p_i = 1 \]
  ○ The system is in state \( |\varphi_i\rangle \) with probability \( p_i \)

Represent missing information and not isolated states
e.g. one qubit of an entangled pair
Dynamics of Mixed States

● Mixed states are represented by **density matrices** $\rho$

\[
\{ (|\varphi_i\rangle, p_i) \} \\
\rho = \sum_i p_i \langle \varphi_i | \varphi_i \rangle
\]

● Isolated System Evolution $\rho \rightarrow \rho'$
  ○ Unitary transformation $\rho' = U \rho U^\dagger$
  ○ Measurement $\rho' = M_m \rho \frac{M_m^\dagger}{\text{tr} (M_m^\dagger M_m \rho)}$

● Open System Evolution $\rho' = \varepsilon (\rho)$
  ○ $\varepsilon$ is a Linear Transformation (super-operator) s.t. …
Super-operators

- Can represent Unitary Operators $U$
  $$\varepsilon (\rho) = U \rho U^\dagger$$
- Can represent the decay for a measurement with result $m$
  $$\varepsilon_m (\rho) = M_m \rho M_m^\dagger$$
- Can represent the decay for a measurement
  $$\varepsilon (\rho) = \sum_m M_m \rho M_m^\dagger$$
- Can represent quantum noises and noisy gates
Open Systems

- Given a composite system $S + E$ in state $\rho_{S+E} \in H_{S+E}$
- The state of the subsystem $S$ is defined as
  $$\rho_S = \text{tr}_E (\rho_{S+E})$$
- And its evolution is according a super-operator $\varepsilon$
Open Systems

Closed Composite System

$\rho_{S+E}(t_0) \xrightarrow{\mathcal{E}_U} \rho_{S+E}(t_1)$

Open Subsystem

$\rho_{S}(t_0) \xrightarrow{\mathcal{E}} \rho_{S}(t_1)$

Crossing the line causes a loss of information: you cannot go back.
Model Checking with Mixed States

- Everything seen so far works with mixed states $\rho$
  - In quantum transition systems:
    - arrows are labeled with super-operators $\varepsilon$
  - In Quantum Logic: $\rho \models A$ iff $\text{supp}(\rho) \subseteq [[A]]$
  - In CTQL: $(1, \rho) \models A$ iff $\text{supp}(\rho) \subseteq [[A]]$

- We can model check noisy circuits!
Optimization via Tensor Networks

A Tensor is a **multidimensional matrix with named indexes**. Formally:

Given a set of indexes $\bar{I} = (i_1, \ldots, i_n)$, a Tensor is a mapping

$$T : \{0, 1\}^{\bar{I}} \rightarrow \mathbb{C}$$
Tensor Representation of Quantum States

- **Single qubit**
  \[ |\varphi\rangle = a |0\rangle + b |1\rangle \]

- **Pair of qubits**
  \[ |\varphi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \]

- **Triplet of qubits**
  \[ |\varphi\rangle = a |000\rangle + b |001\rangle + c |010\rangle + d |011\rangle + e |100\rangle + f |101\rangle + g |110\rangle + h |111\rangle \]
Tensor Representation of Gates

Gates on n qubits can be represented as tensors with indices \( (i_1, \ldots, i_n, i_1', \ldots, i_n') \)

Tensor Network is a hyper-graph with
- Tensors as nodes
- hyper-edges are the shared indexes
Tensor Contraction

A generalization of Matrix product

- $T_1$ on indices $\bar{I}_1 \bar{I}_c$
- $T_2$ on indices $\bar{I}_2 \bar{I}_c$

Contraction returns a Tensor $T'$ on indices $\bar{I}_1 \bar{I}_2$

$$T'_{\bar{I}_1 \bar{I}_2} (\bar{a}, \bar{e}) = \sum_{\bar{o} \in \{0,1\}^c} T_{\bar{I}_1 \bar{I}_c} (\bar{a}, \bar{o}) \cdot T_{\bar{I}_2 \bar{I}_c} (\bar{e}, \bar{o})$$

- **Composing** transformations
- **Applying** transformation to qubits

} In any order
Tensor Contraction

\[ |\varphi\rangle \]
Why Tensor Networks

- Contraction cost depends on the **actual information** stored in the system (linked to entanglement)
- You can choose **any order**
- Thus you can **exploit regularity and locality** in the quantum circuit
Conclusions
Conclusions

- We have seen
  - Quantum Transition Systems
  - CTQL
  - Reduction to CTL model checking
  - Works with Open Systems and noisy gates
  - Optimization via Tensor Networks

- With a small comparison w.r.t.
  - More expressive modeling and logics
Bibliography

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