CONDIZIONI DI OTTIMALITÀ

(P) $\min\{f(x) : x \in X\}$ $\quad f : \mathbb{R}^n \to \mathbb{R}, \quad X \subseteq \mathbb{R}^n$

Teo (condizioni necessarie). Sia $\bar{x} \in X$ un punto di minimo locale di (P).

(i) Se f è differenziabile in $\bar{x}$, allora

$$\langle \nabla f(\bar{x}), d \rangle \geq 0 \qquad \forall d \in T(X,\bar{x}) \tag{1}$$

(ii) Se f è differenziabile 2 volte in $\bar{x}$, allora

$$d \in F(X,\bar{x}),\ \langle \nabla f(\bar{x}), d \rangle = 0 \implies \langle d, \nabla^2 f(\bar{x}) d \rangle \geq 0 \tag{2}$$

dim Sia $\varepsilon > 0$ t.c. $f(\bar{x}) \leq f(x)$ per ogni $x \in X \cap B(\bar{x},\varepsilon)$.

(i) $d \in T(X,\bar{x}) \implies \exists t_k \downarrow 0,\ d_k \to d$ t.c. $\bar{x} + t_k d_k \in X$. Poiché $\bar{x} + t_k d_k \to \bar{x}$, risulta $\bar{x} + t_k d_k \in B(\bar{x},\varepsilon)$ per k suff.nte grande, da cui

$$0 \leq [f(\bar{x}+t_k d_k) - f(\bar{x})]/t_k = \langle \nabla f(\bar{x}), d_k \rangle + r(t_k d_k)/t_k \to \langle \nabla f(\bar{x}), d \rangle$$

e quindi $\langle \nabla f(\bar{x}), d \rangle \geq 0$

(ii) $d \in F(X,\bar{x}) \implies \exists \tau > 0$ t.c. $\bar{x} + td \in X$ per ogni $t \in [0,\tau]$. Quindi per $0 \leq t \leq \min\{\tau,\varepsilon\}$: $0 \leq f(\bar{x}+td) - f(\bar{x}) = t\langle \nabla f(\bar{x}), d \rangle + \frac{1}{2}t^2 \langle d, \nabla^2 f(\bar{x}) d \rangle + r(td)$

$$= \tfrac{1}{2} t^2 \langle d, \nabla^2 f(\bar{x}) d \rangle + r(td)$$

da cui

$$0 \leq \langle d, \nabla^2 f(\bar{x}) d \rangle + 2r(td)/t^2 \xrightarrow[t \downarrow 0]{} \langle d, \nabla^2 f(\bar{x}) d \rangle \qquad \blacksquare$$

Oss (A) $\bar{x} \in \operatorname{int} X$ (in particolare i casi X aperto e $X = \mathbb{R}^n$):

$$(1) \iff \nabla f(\bar{x}) = 0, \qquad (2)+(1) \implies \nabla^2 f(\bar{x}) \text{ semidef. positiva}$$

(B) $(1) \iff -\nabla f(\bar{x}) \in (T(X,\bar{x}))^\circ$

(C) X convesso $\implies T(X,\bar{x}) = \mathrm{cl}\, F(X,\bar{x}) = \mathrm{cl}\ \mathrm{cono}(X - \bar{x})$ da cui

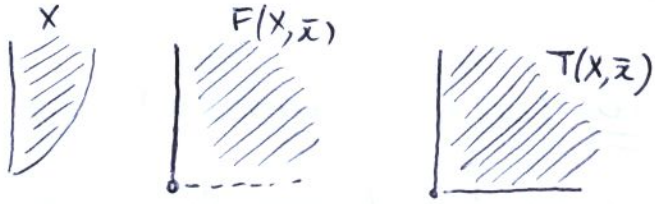
$$(1) \iff \langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in X \iff -\nabla f(\bar{x}) \in N(X,\bar{x})$$

(D) In (ii) non si può sostituire $F(X,\bar{x})$ con $T(X,\bar{x})$:

$n=2$ $\quad f(x)=-x_1^2+x_1x_2+x_2 \quad X=\{x\in\mathbb{R}^2 \mid x_2\geq x_1^2,\ x_1\geq 0\}$

$\bar{x}=(0,0)$ è minimo globale di (P):

$x\in X \Rightarrow f(x)\geq x_1x_2\geq 0=f(\bar{x})$



$\nabla f(\bar{x})=\begin{pmatrix}0\\1\end{pmatrix} \quad \nabla^2 f(\bar{x})=\begin{bmatrix}-2&1\\1&0\end{bmatrix}$

$F(X,\bar{x})=\{d\in\mathbb{R}^2_+ \mid d_2>0\}\cup\{(0,0)\}$

$T(X,\bar{x})=\mathbb{R}^2_+$

$\{d\in F(X,\bar{x}) \mid \langle\nabla f(\bar{x}),d\rangle=0\}=\{(0,0)\}$

$\{d\in T(X,\bar{x}) \mid \langle\nabla f(\bar{x}),d\rangle=0\}=\{(d_1,0) \mid d_1\geq 0\}$

$d=(d_1,0)$ con $d_1\neq 0 \Rightarrow \langle d,\nabla^2 f(\bar{x})d\rangle=-2d_1^2<0$

Nota: (2) non garantisce che $\nabla^2 f(\bar{x})$ sia semidef. positiva (ulteriore esempio: $n=2$, $f(x)=x_1^2-x_2^2$, $X=\{x\in\mathbb{R}^2 \mid x_1\geq 2x_2\geq 0\}$).

Teo (condizione sufficiente) Sia f differenziabile in $\bar{x}$. Allora

$$\langle\nabla f(\bar{x}),d\rangle>0 \quad \forall d\in T(X,\bar{x}),\ d\neq 0 \Rightarrow \bar{x} \text{ punto di minimo locale di (P)}.$$

dim Per assurdo sia $\{x_k\}\subseteq X$ t.c. $x_k\to\bar{x}$ e $f(x_k)<f(\bar{x})$.
Allora $x_k=\bar{x}+t_kd_k$ per $d_k=(x_k-\bar{x})/\|x_k-\bar{x}\|$ e $t_k=\|x_k-\bar{x}\|$. Poiché $\|d_k\|=1$ si può supporre $d_k\to d$ per qualche $d\in\mathbb{R}^n$ con $\|d\|=1$. Risulta per costruzione che $d\in T(X,\bar{x})$ ed inoltre

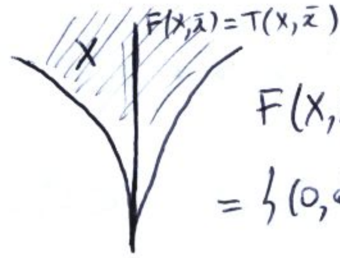
$$0>[f(x_k)-f(\bar{x})]/t_k=\langle\nabla f(\bar{x}),d_k\rangle+r(t_kd_k)/t_k \longrightarrow \langle\nabla f(\bar{x}),d\rangle$$

da cui $\langle\nabla f(\bar{x}),d\rangle\leq 0$ in contraddizione con l'ipotesi $\blacksquare$

Oss Se $\bar{x}\in\operatorname{int}X$, la condizione sufficiente non può essere mai verificata.
A differenza della condizione necessaria (1), la condizione (2) non diventa sufficiente sostituendo $\geq$ con $>$ e neanche richiedendo in aggiunta che valga con $d\in T(X,\bar{x})$

$\underline{\underline{\text{Es}}}$: $n=2$ $\quad f(x) = x_1 + x_2^2$, $\quad X = \{x \in \mathbb{R}^2 \mid x_2^3 \geq x_1^2\}$

$\bar{x} = (0,0)$ soddisfa le condizioni necessarie:



$$F(X,\bar{x}) = T(X,\bar{x}) = \{(0,d_2) \mid d_2 \geq 0\}$$

$$\nabla f(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nabla^2 f(\bar{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$d \in T(X,\bar{x}) \Rightarrow \langle \nabla f(\bar{x}), d \rangle = 0$

$d \in T(X,\bar{x}),\ d \neq 0\ (d_2 > 0) \Rightarrow \langle d, \nabla^2 f(\bar{x}) d \rangle = 2d_2^2 > 0$

$\bar{x} = (0,0)$ non è un minimo locale di (P): $\quad x(t) = (t, \sqrt[3]{t^2}) \in X \quad \forall t \in \mathbb{R}$

$f(x(t)) = t + \sqrt[3]{t^4} = g(t) \qquad g(0) = 0, \ g'(t) = 1 + 4/3 \sqrt[3]{t} \rightarrow g'(0) = 1 > 0$

da cui $g(t)$ per $t<0$ suff.nte vicino a $t_0 = 0$, ovvero $f(x(t)) < f(\bar{x})$

$\underline{\underline{\text{Es}}}$: $X = \mathbb{R}^n$, $\nabla f(\bar{x}) = 0$, $\nabla^2 f(\bar{x})$ def. positiva $\Rightarrow \bar{x}$ minimo locale [stretto] di (P).

CASO CONVESSO

$\boxed{X \subseteq \mathbb{R}^n \text{ convesso}, \quad f: \mathbb{R}^n \to \mathbb{R} \text{ convessa}}$

La dimostrazione del caso (i) delle condizioni necessarie mostra anche

$$f'(\bar{x}; d) \geq 0 \quad \forall d \in T(X,\bar{x}), \quad \text{da cui} \quad \inf_{d \in T(X,\bar{x})} \ \max_{s \in \partial f(\bar{x})} \langle s, d \rangle \geq 0$$

Servirebbe poter invertire inf e max per concludere $\exists s \in \partial f(\bar{x})$ t.c. $-s \in N(X,\bar{x})$.

$\underline{\text{Teo}}$ $\bar{x}$ è un punto di minimo di (P) $\Leftrightarrow 0 \in \partial f(\bar{x}) + N(X,\bar{x})$

$\underline{\underline{\text{dim}}}$ $\Leftarrow$) Sia $s \in \partial f(\bar{x}) \cap -N(X,\bar{x})$:

$$x \in X \Rightarrow f(x) \underset{s \in \partial f(\bar{x})}{\geq} f(\bar{x}) + \langle s, x - \bar{x} \rangle \underset{-s \in N(X,\bar{x})}{\geq} f(\bar{x})$$

$\Rightarrow$) $\quad \bar{x}$ minimo di (P) $\Leftrightarrow (\underbrace{X \cap]-\infty, f(\bar{x})[}_{\text{convesso}}) \cap \underset{\text{convesso}}{\text{epi} f} = \emptyset$

$\rightarrow$

Per il $\underline{\text{teo}}$ (separazione) esistono $(s^*, \mu^*) \neq (0,0) \in \mathbb{R}^n \times \mathbb{R}$, $\gamma \in \mathbb{R}$ t.c.

[*] $\quad < s^*, x > + \mu^* r \leq \gamma \leq < s^*, y > + \mu^* f(y) \quad \forall x \in X, y \in \mathbb{R}^n, r < f(\bar{x})$

Se fosse $\mu^* = 0$, $< s^*, y > \geq \gamma \quad \forall y \in \mathbb{R}^n \Rightarrow s^* = 0$ contraddizione!

Se fosse $\mu^* < 0$ $\quad < s^*, x > + \mu^* r \longrightarrow +\infty$ per $r \to -\infty$ in contraddizione con [*]

Quindi $\mu^* > 0$, e posto $s = -s^*/\mu^*$, considerando $r \to f(\bar{x})$ si ottiene

$$< -s, x > + f(\bar{x}) \leq < -s, y > + f(y) \qquad \forall x \in X, y \in \mathbb{R}^n$$

Scegliendo $x = \bar{x}$, si ha $f(y) \geq f(\bar{x}) + < s, y - \bar{x} > \quad \forall y \in \mathbb{R}^n$

da cui $s \in \partial f(\bar{x})$

Scegliendo $y = \bar{x}$, si ha $< s, x - \bar{x} > \geq 0 \quad \forall x \in X$

da cui $-s \in N(X, \bar{x})$ ∎

$\underline{\text{Oss}}$ $\bar{x} \in \text{int}\, X$: $0 \in \partial f(\bar{x}) + N(X, \bar{x}) \iff 0 \in \partial f(\bar{x})$.

se inoltre f diff. in $\bar{x}$: $0 \in \partial f(\bar{x}) \iff \nabla f(\bar{x}) = 0$.

Chapter 3

Optimality conditions for unconstrained optimization

Given any $f : \mathbb{R}^n \to \mathbb{R}$, optimality conditions for the unconstrained minimization problem

$$(P) \qquad \min\{f(x)\ :\ x \in \mathbb{R}^n\}$$

can be achieved exploiting Taylor's formulas whenever f is differentiable or twice continuously differentiable. The corresponding optimality conditions for unconstrained maximization can be obtained replacing f by $-f$.

3.1 Optimality conditions

Theorem 3.1. *Suppose $\bar{x} \in \mathbb{R}^n$ is a local minimum point of (P).*

(i) If f is differentiable at $\bar{x}$, then $\nabla f(\bar{x}) = 0$;

(ii) If f is twice continuously differentiable at $\bar{x}$, then $\nabla^2 f(\bar{x})$ is positive semidefinite.

Proof. Local optimality guarantees the existence of $\varepsilon > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in B(\bar{x}, \varepsilon)$. Let $d \in \mathbb{R}^n$ be any direction and $t \in]0, \varepsilon[$: $\|d\|_2 = 1$ guarantees $\bar{x} + td \in B(\bar{x}, \varepsilon)$ and therefore $f(\bar{x}) \leq f(\bar{x} + td)$.

(i) Taylor's formula implies

$$0 \leq f(\bar{x} + td) - f(\bar{x}) = t\nabla f(\bar{x})^T d + r_{(f,\bar{x})}(td)$$

and therefore

$$\nabla f(\bar{x})^T d + r_{(f,\bar{x})}(td)/t \geq 0.$$

Since $t = \|td\|_2$, the limit of left-hand side as $t \to 0^+$ provides $\nabla f(\bar{x})^T d \geq 0$. Considering $-d$ the same reasoning provides also $\nabla f(\bar{x})^T d \leq 0$. Thus, $\nabla f(\bar{x})^T d = 0$

holds for any $d \in \mathbb{R}^n$. Taking $d = -\nabla f(\bar{x})$, the equality reads $\|\nabla f(\bar{x})\|_2^2 = 0$ and hence $\nabla f(\bar{x}) = 0$ follows.

(ii) The second-order Taylor's formula (see Theorem 1.7) implies

$$0 \leq f(\bar{x} + td) - f(\bar{x}) = t\nabla f(\bar{x})^T d + \frac{1}{2} t^2 d^T \nabla^2 f(\bar{x}) d + r_{(f,\bar{x})}(td).$$

Since (i) guarantees $\nabla f(\bar{x}) = 0$, then

$$d^T \nabla^2 f(\bar{x}) d + r_{(f,\bar{x})}(td)/2t^2 \geq 0$$

holds too. Since $t^2 = \|td\|_2^2$, the limit of the left-hand side as $t \to 0$ provides the inequality $d^T \nabla^2 f(\bar{x}) d \geq 0$. Since d is an arbitrary direction, $\nabla^2 f(\bar{x})$ is positive semidefinite. $\square$

If $\bar{x} \in \operatorname{int} D$ minimizes f over some $D \subseteq \mathbb{R}^n$, then the necessary conditions of Theorem 3.1 hold also in this case: the above proof still works just considering any $\varepsilon > 0$ which in addition satisfies $B(\bar{x}, \varepsilon) \subseteq D$.

Definition 3.1. $\bar{x} \in \mathbb{R}^n$ is called a *stationary point of* f if $\nabla f(\bar{x}) = 0$.

Looking for stationary points of f amounts to solving the system of n equations

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, ..., x_n) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, ..., x_n) = 0 \end{cases}$$

in the n unknowns $(x_1, \ldots, x_n)$. This is generally a nonlinear system, but if the quadratic function $f(x) = \frac{1}{2} x^T Q x + b^T x + c$ is considered then it is actually the linear system $Qx = -b$ (since $\nabla f(x) = Qx + b$). If f is strictly convex, then $\nabla^2 f(x) \equiv Q$ is positive definite and therefore invertible: $\bar{x} = -Q^{-1} b$ is the unique stationary point and it is the unique minimum point (see Theorems 3.2 and 3.3 below). On the contrary, if f is not convex, due to Theorem 3.1(ii) no stationary point is a local minimum since Q is not positive semidefinite.

Example 3.1. Take $n = 2$ and $f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 4x_1^2)$:

$$\nabla f(x) = \begin{pmatrix} 16x_1^3 - 10x_1 x_2 \\ 2x_2 - 5x_1^2 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} 48x_1^2 - 10x_2 & -10x_1 \\ -10x_1 & 2 \end{bmatrix}.$$

Then, $\nabla f(x) = 0$ if and only if $x = (0, 0)$ and moreover

$$\nabla^2 f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

is positive semidefinite (but not definite). Anyway, $(0, 0)$ is not a local minimum point of (P). In fact,

$$f(x_1, 2x_1^2) = -2x_1^2 < 0$$

for any $x_1 \neq 0$. Therefore, f is negative along the parabola $\{x \in \mathbb{R}^2 \ : \ x_2 = 2x_1^2\}$. Notice that f is not even a local maximum point of (P) : $\nabla^2 f(0,0)$ is not negative semidefinite and in fact f is positive along all the parabolas $\{x \in \mathbb{R}^2 \ : \ x_2 = \alpha x_1^2\}$ with $\alpha > 4$.

Theorem 3.2. *Let f be twice continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a stationary point of f such that $\nabla^2 f(\bar{x})$ is positive definite, then it is a strict local minimum point of (P) and moreover there exist $\delta, \gamma > 0$ such that*

$$\forall\ x \in B(\bar{x}, \delta) \ : \ f(x) \geq f(\bar{x}) + \gamma \|x - \bar{x}\|_2^2.$$

Proof. It is enough to prove the above inequality as it guarantees strict local optimality too. Taking any $x \in \mathbb{R}^n$, the second-order Taylor's formula (see Theorem 1.7) implies

$$\begin{aligned} f(x) - f(\bar{x}) &= \nabla f(\bar{x})^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) + r_{(f,\bar{x})}(x - \bar{x}) \\ &= \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}) + r_{(f,\bar{x})}(x - \bar{x}) \\ &\geq \frac{1}{2}\lambda_{min} \|x - \bar{x}\|_2^2 + r_{(f,\bar{x})}(x - \bar{x}) \end{aligned}$$

and therefore

$$[f(x) - f(\bar{x})]/\|x - \bar{x}\|_2^2 \geq \lambda_{min}/2 + r_{(f,\bar{x})}(x - \bar{x})/\|x - \bar{x}\|_2^2$$

where $\lambda_{min} > 0$ is the minimum eigenvalue of $\nabla^2 f(\bar{x})$.[1] Choose any positive threshold $\varepsilon < \lambda_{min}/2$. Since the limit of the right-hand side as $x \to \bar{x}$ is $\lambda_{min}/2$, there exists $\delta > 0$ such that

$$\forall\ x \in B(\bar{x}, \delta) \ : \ [f(x) - f(\bar{x})]/\|x - \bar{x}\|_2^2 \geq (\lambda_{min}/2 - \varepsilon).$$

Setting $\gamma = \lambda_{min}/2 - \varepsilon$, the thesis follows from the above inequality. □

If f is a strictly convex quadratic function, then the above theorem holds with $\gamma = \lambda_{min}/2$ (where λ_{min} is the minimum eigenvalue of Q) and any $\delta > 0$. In fact, $\bar{x} = -Q^{-1}b$ is the unique stationary point of f and

$$\forall\ x \in \mathbb{R}^n \ : \ f(x) - f(\bar{x}) = \frac{1}{2}(x - \bar{x})^T Q (x - \bar{x}).$$

3.2 Optimality conditions in the convex case

Theorem 3.3. *Let f be convex and differentiable (on $\mathbb{R}^n$). Then, $\bar{x} \in \mathbb{R}^n$ is a minimum point of (P) if and only if $\nabla f(\bar{x}) = 0$.*

[1] Given any symmetric matrix $Q \in \mathbb{R}^{n \times n}$ the inequality $y^T Q y \geq \lambda_{min} \|y\|_2^2$ holds for any $y \in \mathbb{R}^n$ where λ_{min} denotes the minimum eigenvalue of Q

Proof. *Only if)* It is just Theorem 3.1(i).

If) By Theorem 2.3 the convexity of f guarantees

$$f(y) \geq f(\bar{x}) + \nabla f(\bar{x})^T (y - \bar{x})$$

for any $y \in \mathbb{R}^n$. Since $\nabla f(\bar{x}) = 0$, the optimality of $\bar{x}$ follows immediately. □

Notice that any (twice continuously differentiable) convex function f satisfies the second-order optimality condition of Theorem 3.1 at any point (see Theorem 2.5). Moreover, it does not have any global maximum point unless it is a constant function: in fact, a maximum point is a stationary point (just apply Theorem 3.1 to $-f$) and hence it is also a minimum point by Theorem 3.3. The same reasoning applies to local maximum points, which may exist if they are actually also minimum points.

The minimum points of the convex quadratic function $f(x) = \frac{1}{2}x^T Qx + b^T x + c$ are the solutions of the linear system $Qx + b = 0$. If Q is positive definite, then $-Q^{-1}b$ is the unique minimum point. If Q is positive semidefinite but not positive definite, there are infinitely many minimum points if at least one exists but f could be unbounded by below.

Proposition 3.1. *Let $f(x) = \frac{1}{2}x^T Qx + b^T x + c$ be convex. Then, f is unbounded by below if and only if there exists $\hat{x} \in \mathbb{R}^n$ such that $Q\hat{x} = 0$ and $b^T\hat{x} \neq 0$.*

Proof. *If)* Take $x(t) = t\hat{x}$. If $b^T\hat{x} > 0$ (< 0), then

$$f(x(t)) = t(b^T\hat{x}) + c \to -\infty \quad \text{as} \quad t \to -\infty \ (+\infty)$$

Only if) Since Q is symmetric, there exists an orthonormal basis $\{x^1, \ldots, x^n\}$ of $\mathbb{R}^n$ composed by eigenvectors of Q, that is ${x^i}^T x^j = 0$ for all $i \neq j$ and $Qx^i = \lambda_i x^i$ for all $i = 1, \ldots, n$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Q. Given any $x \in \mathbb{R}^n$, there exist $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} \gamma_i x^i$. Therefore,

$$f(x) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i \gamma_i^2 + \sum_{i=1}^{n} (b^T x^i)\gamma_i = \sum_{i=1}^{n} [\frac{1}{2}\lambda_i \gamma_i^2 + (b^T x^i)\gamma_i].$$

Ab absurdo, suppose $b^T x = 0$ whenever $Qx = 0$, which implies that $b^T x^i = 0$ if $\lambda_i = 0$. Therefore, each nonzero term in the above sum gets its minimum value for $\gamma_i = \bar{\gamma}_i = -b^T x^i \lambda_i$, and f is bounded by below since

$$f(x) \geq \sum_{i \in I} [\frac{1}{2}\lambda_i \bar{\gamma}_i^2 + (b^T x^i)\bar{\gamma}_i] = -\sum_{i \in I} (b^T x^i)^2 / 2\lambda_i$$

where $I = \{i \ : |\lambda_i \neq 0\}$. □

$X = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0 ,\ h_j(x) = 0 ,\ i \in I,\ j \in J \}$ $\quad I = \{1, \dots, p\}$ $\ J = \{1, \dots, q\}$

$I(\bar{x}) = \{ i \in I \mid g_i(\bar{x}) = 0 \}$, $\ g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ diff. con continuità, $\bar{x} \in X$

$g_i(x) \approx g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle$, $\quad h_j(x) \approx h_j(\bar{x}) + \langle \nabla h_j(\bar{x}), x - \bar{x} \rangle$

$L_{\leq}(X, \bar{x}) = \{ d \in \mathbb{R}^n \mid \langle \nabla g_i(\bar{x}), d \rangle \leq 0 ,\ i \in I(\bar{x}),\ \langle \nabla h_j(\bar{x}), d \rangle = 0 \}$

<u>**Prop**</u> (i) $T(X, \bar{x}) \subseteq L_{\leq}(X, \bar{x})$ $\quad$ (ii) $\{ \nabla h_j(\bar{x}) \}_{j \in J}$ lin. indip. $\Rightarrow L_{<}(X, \bar{x}) \subseteq T(X, \bar{x})$

<u>**dim**</u> (i) Sia $d \in T(X, \bar{x})$: $\exists\, t_n \downarrow 0,\ d_n \to d$ t.c. $\bar{x} + t_n d_n \in X$. Sia $i \in I(\bar{x})$:

$$0 \geq [g_i(\bar{x} + t_n d_n) - g_i(\bar{x})]/t_n = \langle \nabla g_i(\bar{x}), d_n \rangle + r(t_n d_n)/t_n \longrightarrow \langle \nabla g_i(\bar{x}), d \rangle$$

Analogamente $j \in J$: $\ 0 = [h_j(\bar{x} + t_n d_n) - h_j(\bar{x})]/t_n \longrightarrow \langle \nabla h_j(\bar{x}), d \rangle$.

(ii) Sia $h = (h_1, \dots, h_q) : \mathbb{R}^n \to \mathbb{R}^q$: $h(\bar{x}) = 0$, $J_h(\bar{x}) = \begin{bmatrix} - \nabla h_1(\bar{x})^T - \\ \vdots \\ - \nabla h_q(\bar{x})^T - \end{bmatrix}$

$J_h(\bar{x}) \in \mathbb{R}^{q \times n}$ ha rango massimo $= q$ per l'ipotesi su $\{\nabla h_j(\bar{x})\}_{j \in J}$

Sia $d \in L_<(X, \bar{x})$: $\langle \nabla h_j(\bar{x}), d \rangle = 0 \Rightarrow J_h(\bar{x}) d = 0$

Sia $K : \mathbb{R}^{1+q} \to \mathbb{R}^q$ data da $K(t, u) = h(\bar{x} + td + J_h(\bar{x})^T u)$: $K(0,0) = h(\bar{x}) = 0$

e $(J_K)_u(0,0) = J_h(\bar{x}) J_h(\bar{x})^T \in \mathbb{R}^{q \times q}$ è definita positiva (e quindi invertibile) in quanto

$J_h(\bar{x})$ ha rango max : $\langle v, (J_K)_u(0,0) v \rangle = \langle J_h(\bar{x})^T v, J_h(\bar{x})^T v \rangle = \| J_h(\bar{x})^T v \|^2 > 0$

$(v \neq 0 \Rightarrow J_h(\bar{x})^T v \neq 0)$

Per il teo (funzione implicita - Dini) esistono $\delta > 0$ e

$u :]-\delta, \delta[\to \mathbb{R}^q$ diff. con continuità t.c. $K(t, u(t)) = 0$ $\forall t \in]-\delta, \delta[$ e $u(0) = 0$

Sia $x(t) = \bar{x} + td + J_h(\bar{x})^T u(t)$: $h(x(t)) = 0$ $\forall t \in]-\delta, \delta[$ garantisce

$$0 = \frac{d}{dt} h(x(t))\Big|_{t=0} = J_h(\bar{x}) \left[d + J_h(\bar{x})^T u'(0) \right] = J_h(\bar{x}) J_h(\bar{x})^T u'(0) \text{ da cui } u'(0) = 0$$

dove $u'(0) = (u_1'(0), \dots, u_q'(0))$. Pertanto $x'(0) = d$, $x(t) = \bar{x} + t d_t$ con

$d_t \to d$ per $t \to 0$, con $h(\bar{x}+td_t)=0$. In particolare $\forall t_n \downarrow 0$ $d_n = d_{t_n} \to d$

e $h_j(\bar{x}+t_n d_n)=0$ $j \in J$.

$i \notin I(\bar{x})$: $g_i(\bar{x}+t_n d_n) \longrightarrow g_i(\bar{x})<0$, da cui $g_i(\bar{x}+t_n d_n)<0$ se n è suff.nte grande

$i \in I(\bar{x})$: $g_i(\bar{x}+t_n d_n)/t_n = g_i(\bar{x})/t_n + \langle \nabla g_i(\bar{x}), d_n \rangle + r(t_n d_n)/t_n =$

$d \in L_<(X,\bar{x})$

$$= \langle \nabla g_i(\bar{x}), d_n \rangle + r(t_n d_n)/t_n \longrightarrow \langle \nabla g_i(\bar{x}), d \rangle < 0$$

da cui $g_i(\bar{x}+t_n d_n)<0$ se n è suff.nte grande.

In conclusione $\bar{x}+t_n d_n \in X$ se n è suff.nte grande, e quindi $d \in T(X,\bar{x})$ ■

Sia H il sottospazio affine generato da $\{ d \in \mathbb{R}^n \mid \langle \nabla h_j(\bar{x}), d \rangle = 0 \quad j \in J \}$

$L_<(X,\bar{x}) \subseteq H$ è aperto in H

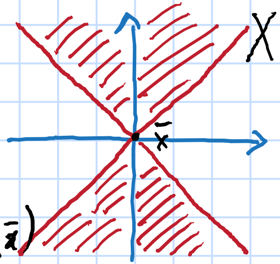
$T(X,\bar{x}) \subseteq H$ è chiuso

$\Rightarrow L_<(X,\bar{x}) \neq T(X,\bar{x})$

(a meno che $L_<(X,\bar{x}) = H = T(X,\bar{x})$)

Es: $X=\{x\in\mathbb{R}^2 \mid x_1^2-x_2^2\le 0\}$, $\bar{x}=(0,0)$: $T(X,\bar{x})=X$

$g_1(x)=x_1^2-x_2^2$, $\nabla g_1(0,0)=(0,0) \to L_<(X,\bar{x})=\emptyset$, $L_\le(X,\bar{x})=\mathbb{R}^2$



$\bar{x}\in X$ punto di minimo locale di (P) $\Rightarrow$ $\langle\nabla f(\bar{x}),d\rangle\ge 0 \quad \forall d\in T(X,\bar{x})$

$$\equiv$$

$$\begin{cases}\langle\nabla f(\bar{x}),d\rangle<0\\ d\in L_<(X,\bar{x})\end{cases} \text{ non ammette soluzione } d\in\mathbb{R}^n \xLeftarrow{\{\nabla h_j(\bar{x})\} \text{ lin. indip.}} \begin{cases}\langle\nabla f(\bar{x}),d\rangle<0\\ d\in T(X,\bar{x})\end{cases} \text{ non ammette soluzione } d\in\mathbb{R}^n$$

$$\equiv$$

$$\begin{cases}\langle\nabla f(\bar{x}),d\rangle<0\\ \langle\nabla g_i(\bar{x}),d\rangle<0 & i\in I(\bar{x})\\ \langle\nabla h_j(\bar{x}),d\rangle=0 & j\in J\end{cases} \text{ non ammette soluzione } d\in\mathbb{R}^n \overset{\text{Motzkin}}{\equiv} \exists\, \theta,\lambda_i\ge 0 \text{ non tutti nulli}, \mu_j\in\mathbb{R} \text{ t.c.}$$

$$\boxed{\theta\nabla f(\bar{x})+\sum_{i\in I(\bar{x})}\lambda_i\nabla g_i(\bar{x})+\sum_{j\in J}\mu_j\nabla h_j(\bar{x})=0}$$

REGOLA DEI MOLTIPLICATORI

Oss $\{\nabla h_j(\bar{x})\}_{j\in J}$ lin. dip. $\Rightarrow$ regola vale con $\theta=\lambda_i=0$ e μ_j opportuni.

Ponendo $\lambda_i = 0$ per $i \notin I(\bar{x})$ si ottiene

Teo (Fritz-John) Sia $\bar{x} \in X$ un punto di minimo locale di (P). Allora esistono $\theta \geq 0$, $\lambda_i \geq 0$, $\mu_j \in \mathbb{R}$ $(i \in I, j \in J)$ non tutti nulli tali che

$$\theta \nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} \mu_j \nabla h_j(\bar{x}) = 0$$

$$\lambda_i g_i(\bar{x}) = 0 \qquad i \in I. \qquad \text{(FJ)}$$

Oss $\nabla f(\bar{x}) = 0 \Rightarrow$ (FJ) valgono con $\theta = 1$, $\lambda_i = \mu_j = 0$

$$[\text{(FJ) con } \theta \neq 0] \equiv \begin{cases} \langle \nabla f(\bar{x}), d \rangle < 0 \\ \langle \nabla g_i(\bar{x}), d \rangle \leq 0 \\ \langle \nabla h_j(\bar{x}), d \rangle = 0 \end{cases} \text{non ammette soluzione } d \in \mathbb{R}^n \equiv \begin{cases} \langle \nabla f(\bar{x}), d \rangle < 0 \\ d \in L_{\leq}(X, \bar{x}) \end{cases} \text{non ammette soluzione } d \in \mathbb{R}^n$$

$\begin{cases} \langle \nabla f(\bar{x}), d \rangle < 0 \\ d \in L_{\leq}(X, \bar{x}) \end{cases}$ non ammette soluzione $\Longleftrightarrow$ $\begin{cases} \langle \nabla f(\bar{x}), d \rangle < 0 \\ d \in T(X, \bar{x}) \end{cases}$ non ammette soluzione $\Longleftarrow$ $\bar{x} \in X$ punto di minimo locale di (P)

se $T(X, \bar{x}) = L_{\leq}(X, \bar{x})$ $\longleftarrow$ condizione di Abadie

Oss (i) $L_{\leq}(X, \bar{x})$ è convesso e chiuso: $\operatorname{cl} \operatorname{conv} T(X, \bar{x}) \subseteq L_{\leq}(X, \bar{x})$

(ii) $\langle \nabla f(\bar{x}), d \rangle \geq 0 \ \forall d \in T(X, \bar{x}) \iff \langle \nabla f(\bar{x}), d \rangle \geq 0 \ \forall d \in \operatorname{cl} \operatorname{conv} T(X, \bar{x})$

Teo (Karush-Kuhn-Tucker) Sia $\bar{x} \in X$ un punto di minimo locale di (P) tale che $\operatorname{cl} \operatorname{conv} T(X, \bar{x}) = L_{\leq}(X, \bar{x})$. Allora esistono $\lambda_i \geq 0$, $\mu_j \in \mathbb{R}$ tali che

condizione di Guignard

$$\nabla f(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} \mu_j \nabla h_j(\bar{x}) = 0 \qquad \text{(KKT)}$$

condizioni di complementarità $\longrightarrow$ $\lambda_i g_i(\bar{x}) = 0 \quad i \in I$

moltiplicatori di Lagrange

QUALIFICA dei VINCOLI : condizione su $\underset{(g_i, h_j)}{X}$ per cui cl conv $T(X,\bar{x}) = L_{\leq}(X,\bar{x})$

LINEARE INDIPENDENZA (LI) : $\{\nabla g_i(\bar{x})\}_{i \in I(\bar{x})} \cup \{\nabla h_j(\bar{x})\}_{j \in J}$ lin. indip.

SLATER (SL) : $\underset{i \in I(\bar{x})}{g_i}$ convesse, h_j affini, $\exists \hat{x} \in \mathbb{R}^n$ t.c. $\begin{cases} g_i(\hat{x}) < 0 & i \in I(\bar{x}) \\ h_j(\hat{x}) = 0 & j \in J \end{cases}$

MANGASARIAN-FROMOVITZ (MF) : $\{\nabla h_j(\bar{x})\}_{j \in J}$ lin. indip., $L_{<}(X,\bar{x}) \neq \emptyset$

Prop (i) (MF) vale in $\bar{x}$ $\Rightarrow$ $T(X,\bar{x}) = L_{\leq}(X,\bar{x})$

(ii) (LI) $\Rightarrow$ (MF) (iii) (SL) $\Rightarrow$ (MF)

dim (i) $T(X,\bar{x})$ chiuso $\Rightarrow$ cl $L_{<}(X,\bar{x}) \subseteq T(X,\bar{x}) \subseteq L_{\leq}(X,\bar{x})$

Siano $\bar{d} \in L_{<}(X,\bar{x})$, $d \in L_{\leq}(X,\bar{x})$: $d_n = d + 1/n\,\bar{d} \in L_{<}(X,\bar{x})$

Infatti: $\langle \nabla h_j(\bar{x}), d_n \rangle = \langle \nabla h_j(\bar{x}), d \rangle + 1/n \langle \nabla h_j(\bar{x}), \bar{d} \rangle = 0$

$$< \nabla g_i(\bar{x}), d_n > = < \nabla g_i(\bar{x}), d > + 1/n < \nabla g_i(\bar{x}), \bar{d} > < < \nabla g_i(\bar{x}), d > \leq 0$$

Pertanto $d_n \to d$ garantisce $d \in \mathrm{cl}\, L_<(X, \bar{x})$, da cui $\mathrm{cl}\, L_<(X, \bar{x}) = T(X, \bar{x}) = L_\leq(X, \bar{x})$

(ii) Se (MF) non vale, allora $\{\nabla h_j(\bar{x})\}_{j \in J}$ sono lin. dip. oppure $L_<(X, \bar{x}) = \emptyset$.
Nel primo caso ovviamente anche (LI) non vale. Per il teo (Motzkin) $L_<(X, \bar{x}) = \emptyset$ equivale alla lin. dip. di $\{\nabla g_i(\bar{x})\}_{i \in I(\bar{x})} \cup \{\nabla h_j(\bar{x})\}_{j \in J}$

(iii) Poiché h_j sono affini, possiamo supporre $q \leq n$ e $\{\nabla h_j(\bar{x})\}_{j \in J}$ lin. indip.
(i vincoli $h_j(x) = 0$ individuano iperpiani affini)

g_i convessa

$$0 > g_i(\hat{x}) \geq g_i(\bar{x}) + < \nabla g_i(\bar{x}), x - \bar{x} > = < \nabla g_i(\bar{x}), x - \bar{x} > \quad \text{se } i \in I(\bar{x})$$

$h_j(x) = < a_j, x > + b_j$ per opportuni $a_j \in \mathbb{R}^n$, $b_j \in \mathbb{R}$:

$$< \nabla h_j(\bar{x}), \hat{x} - \bar{x} > = < a_j, \hat{x} > - < a_j, \bar{x} > = b_j - b_j = 0$$

Quindi $(\hat{x} - \bar{x}) \in L_<(X, \bar{x})$ ∎

Sia $M(\bar{x}) = \{ (\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \mid (\bar{x}, \lambda, \mu)$ soddisfano le condizioni (KKT) $\}$.

<u>Prop</u> (i) Supponiamo $M(\bar{x}) \neq \emptyset$. Allora $|M(\bar{x})| = 1 \iff$ (LI) vale in $\bar{x}$.

(ii) $M(\bar{x})$ è compatto $\iff$ (MF) vale in $\bar{x}$

<u>dim</u> (i) $\boxed{\sum_{i \in I(\bar{x})} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j \in J} \bar{\mu}_j \nabla h_j(\bar{x}) = 0 \quad (1)}$ per $\bar{\lambda}_i \geq 0, \bar{\mu}_j \in \mathbb{R}$ non tutti nulli

$\Leftarrow$) Siano $(\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in M(\bar{x})$. Sottraendo ora regola dei moltiplicatori dall' altra si ottiene $\sum_{i \in I(\bar{x})} (\lambda_i - \bar{\lambda}_i) \nabla g_i(\bar{x}) + \sum_{j \in J} (\mu_j - \bar{\mu}_j) \nabla h_j(\bar{x}) = 0$, da cui $\lambda_i = \bar{\lambda}_i$, $\mu_j = \bar{\mu}_j$ per ogni $i \in I(\bar{x}), j \in J$ per l'ipotesi di lineare indipendenza.

(ii) $\Rightarrow$) Se (MF) non vale, allora <u>teo</u> (Motzkin) o la lin. dipendenza di $\{\nabla h_j(\bar{x})\}_{j \in J}$ garantiscono che (1) vale per opportuni $\bar{\lambda}_i \geq 0$, $\bar{\mu}_j \in \mathbb{R}$ non tutti nulli. Sia $(\lambda,\mu)\in M(\bar{x})$: $m(t)=(\lambda+t\bar{\lambda},\mu+t\bar{\mu})\in M(\bar{x})$ $\forall t$ e $\|m(t)\|\to+\infty$ se $t\to\pm\infty$

$\Leftarrow$) Supponiamo $\exists\{(\lambda^k,\mu^k)\}\subseteq M(\bar{x})$ con $\gamma_k=\|(\lambda^k,\mu^k)\|\to+\infty$.

Considerando eventualmente l'opportuna sottosuccessione, possiamo supporre $\gamma_k^{-1}\lambda^k\to\bar{\lambda}$, $\gamma_k^{-1}\mu^k\to\bar{\mu}$ per qualche $\bar{\lambda}\in\mathbb{R}^p_+$, $\bar{\mu}\in\mathbb{R}^q$ con $\|(\bar{\lambda},\bar{\mu})\|=1$. Abbiamo inoltre $\gamma_k^{-1}\nabla f(\bar{x})+\sum_{i=1}^{p}\gamma_k^{-1}(\lambda^k)_i\nabla g_i(\bar{x})+\sum_{j=1}^{q}\gamma_k^{-1}(\mu^k)_j\nabla h_j(\bar{x})=0$ e passando al limite: $\sum_{i=1}^{p}\bar{\lambda}_i\nabla g_i(\bar{x})+\sum_{j=1}^{q}\bar{\mu}_j\nabla h_j(\bar{x})=0$. Se $\bar{\lambda}=0$, allora $\bar{\mu}\neq0$ e $\{\nabla h_j(\bar{x})\}_{j\in J}$ sono lin. dip. Se $\bar{\lambda}\neq0$, il teo (Motzkin) garantisce $L_<(X,\bar{x})=\emptyset$.

In entrambi i casi (MF) non vale. ■

SUFFICIENZA DELLE CONDIZIONI KKT

Teo Siano f e g_i convesse con $i\in I(\bar{x})$ per qualche $\bar{x}\in X$, siano h_j

affini ($j \in J$). Se esistono $\lambda \in \mathbb{R}^p_+$, $\mu \in \mathbb{R}^q$ tali che $(\bar{x}, \lambda, \mu)$ soddisfa le condizioni KKT, allora $\bar{x}$ è un punto di minimo globale di (P).

<u>dim</u> Sia $x \in X$:

$$f(x) - f(\bar{x}) \overset{f \text{ convessa}}{\geq} \langle \nabla f(\bar{x}), x - \bar{x} \rangle \overset{KKT}{=} - \sum_{i \in I(\bar{x})} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle - \sum_{j \in J} \mu_j \langle \nabla h_j(\bar{x}), x - \bar{x} \rangle$$

$$\underset{g_i \text{ convessa}}{\geq} - \sum_{i \in I(\bar{x})} \lambda_i (g_i(x) - g_i(\bar{x})) - \sum_{j \in J} \mu_j h_j(x) = - \sum_{i \in I(\bar{x})} \lambda_i g_i(x) \geq 0 \qquad \blacksquare$$

g_i convessa

h_j affine $\rightarrow h_j(x) = h_j(\bar{x}) + \langle \nabla h_j(\bar{x}), x - \bar{x} \rangle = \langle \nabla h_j(\bar{x}), x - \bar{x} \rangle$